J. W. Hale.

A TREATISE ON ALGEBRA,

FOR THE USE OF

SCHOOLS AND COLLEGES.

BY

S. CHASE,

PROFESSOR OF MATHEMATICS IN DARTMOUTH COLLEGE.

NEW-YORK:
D. APPLETON & CO., 200 BROADWAY.
PHILADELPHIA:
GEO. S. APPLETON, 164 CHESNUT-ST.

MDCCCXLIX.

DARTMOUTH PRESS,
Hanover, N. H.

PREFACE.

The following treatise is intended to exhibit such a view of the principles of Algebra, as shall best prepare the student for the further pursuit of mathematical studies.

The principles presented I have endeavored to enunciate as clearly and briefly as possible, to demonstrate rigorously, and to illustrate by strictly pertinent examples.

Part of the examples are of the most elementary form, part are purely numerical, and a large part of the rest are expressions employed in the reasonings and investigations of Trigonometry, Analytical Geometry, Mechanics and other branches of mathematical study. Thus, the application of the principle is exhibited, relieved of all extraneous difficulty, and connected with the familiar ideas of Arithmetic; and, moreover, the forms and operations employed in demonstrating truths of Geometry, and of other related sciences, are rendered familiar, and made ready for use when they shall be needed.

This last consideration is of great importance. Much of the difficulty which students find in later parts of the course results from want of familiarity with the algebraic expressions employed, and with the elementary operations performed upon them. At the same time, such expressions and operations are frequently among the most convenient illustrations of algebraic principles.

The discussion of the theory of exponents and powers (§§ 11–24) is, so far as I know, original. The use and interpre-

tation of the fractional and negative exponents is exhibited as a necessary consequence of the definition.

The demonstration of the Binomial Theorem for negative and fractional exponents (§§ 291–294), and the development of the fundamental logarithmic formula (§§ 320–323) are substantially those of Lagrange.

The nature of the *Modulus* (§§ 327–332), and some of the properties of logarithmic differences (§§ 333–336) are discussed more fully than I have seen them in any elementary treatise. Familiarity with these principles is of great advantage to the student, and their discussion is, by no means, difficult.

A table of the principal formulæ of the book is placed after the table of contents, for convenience of reference and review. It has also the advantage of generalizing, and bringing into one view, principles exhibited, with more or less fulness, in different parts of the book. For the suggestion of this table, I am indebted to Mr. Richards, the able Principal of Kimball Union Academy.

I am also very greatly indebted to my associates, Professors Crosby and Young, for valuable suggestions and criticisms. In correcting the proofs of the last half of the work, I have had the assistance of Mr. Edward Webster, a recent graduate of the College, whose tastes and attainments qualify him to do excellent service in the cause of science.

S. C.

Dartmouth College, May 1, 1849.

CONTENTS.

INTRODUCTION.

CHAPTER I.

ADDITION AND SUBTRACTION.

CHAPTER II.

MULTIPLICATION AND DIVISION.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

CHAPTER XI.

CHAPTER XII.

CHAPTER XIII.

DIFFERENCES.

CHAPTER XIV.

CHAPTER XV.

LOGARITHMS.

CHAPTER XVI.

THEORY OF EQUATIONS.

CHAPTER XVII.

FORMULÆ.

§ 6. *a.*) $+a > 0$; $-a < 0$. That is,

A *positive* quantity > 0; a *negative* quantity < 0.

§ 7. *a, b.*) $-(-a) = +a$. $\therefore -[-(-a)] = -a$;

$-(-(-(-a))) = +a$; &c. §§ 63; 68. *a, c.*

§ 13. $a^0 = 1$. § 17. $a^{-n} = \frac{1}{a^n}$.

§ 23. $\sqrt[n]{a} = a^{\frac{1}{n}}$. $\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{\frac{m}{n}}$. §§ 12, 25.

§ 57. 3. $\frac{1}{2}(a+b) + \frac{1}{2}(a-b) = a$.

§ 60. 4. $\frac{1}{2}(a+b) - \frac{1}{2}(a-b) = b$.

§§ 89, 90. $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

§ 91. $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$.

$(a+b)^2 - (a-b)^2 = 4ab$.

§ 92. $(a+b)(a-b) = a^2 - b^2$.

§ 96. *a.*) $\frac{a^n - b^n}{a-b} = a^{n-1} + a^{n-2}b \ldots + ab^{n-2} + b^{n-1}$.

b.) $\frac{a^n - a^n}{a-a} = na^{n-1}$.

§ 97. $\frac{a^{2n} - b^{2n}}{a+b} = a^{2n-1} - a^{2n-2}b + \ldots + ab^{2n-2} - b^{2n-1}$.

§ 98. $\frac{a^{2n+1} + b^{2n+1}}{a+b} = a^{2n} - a^{2n-1}b + \ldots - ab^{2n-1} + b^{2n}$.

§§ 109, 139, 140.) $\frac{a}{0} = \infty$. $\frac{a}{\infty} = 0$. $\frac{0}{0}$, *indeterminate.*

§ 151, *c.*) $(a^n)^m = a^{mn}$. $(\pm a)^{2n} = +(a^{2n})$.

$(\pm a)^{2n+1} = \pm(a^{2n+1})$.

§ 152. $(+a)^{\frac{1}{2n}} = \pm(a^{\frac{1}{2n}})$. $(-a)^{\frac{1}{2n}}$, *imaginary.*

$(\pm a)^{\frac{1}{2n+1}} = \pm(a^{\frac{1}{2n+1}})$.

§ 158. $(-a^2)^{\frac{1}{2}} = a(-1)^{\frac{1}{2}} = a\sqrt{-1}$.

§ 162. $(\sqrt{-1})^2 = -1$; $(\sqrt{-1})^3 = -\sqrt{-1}$.

$(\sqrt{-1})^4 = 1$; $(\sqrt{-1})^5 = \sqrt{-1}$.

§ 185. $(a \pm b^{\frac{1}{2}})^{\frac{1}{2}} = \left(\frac{a+c}{2}\right)^{\frac{1}{2}} \pm \left(\frac{a-c}{2}\right)^{\frac{1}{2}}$;

where $c = (a^2 - b)^{\frac{1}{2}}$.

§ 186. $(a + b^{\frac{1}{2}})(a - b^{\frac{1}{2}}) = a^2 - b$.

§ 207. $x^2 + 2px + q^2 = 0 = (x - a_1)(x - a_2)$.

$2p = -(a_1 + a_2)$. $q^2 = a_1 a_2$.

$x = -p \pm \sqrt{(p^2 - q^2)}$.

§§ 232, 233. If $a : b = k : l$, then $al = bk$;

§ 234. $a : k = b : l$; $l : b = k : a$;

§ 235. $b : a = l : k$;

§ 236. $a \pm b : a = k \pm l : k$;

§ 238. $a \pm nb : k \pm nl = b \pm ma : l \pm mk$;

§ 239. $ma : nb = mk : nl$;

§ 241. $a^n : b^n = k^n : l^n$.

§ 240. If $a : b = e : f = g : h = k : l$,

then $a + e + g + k : b + f + h + l = a : b$.

§ 242. $a : b = k : l$; $e : f = g : h$; $r : s = x : y$.

$\therefore$ $aer : bfs = kgx : lhy$.

§§ 250, 251. $l = a + (n-1)D$. $s = \frac{1}{2}n(a+l)$.

§§ 258–261. $l = am^{n-1}$; $s = \dfrac{a(m^n-1)}{m-1} = \dfrac{lm-a}{m-1}$.

§ 258. $A = p(1+r)^t$. $p = \dfrac{A}{(1+r)^t}$.

§ 262. $A' = a\dfrac{(1+r)^t-1}{r}$. $p' = \dfrac{a}{r}\left(1 - \dfrac{1}{(1+r)^t}\right)$.

h.) $A = p\left\{1+\dfrac{r}{n}\right\}^{nt}$

$$= p\left\{1 + \frac{nt}{1}\cdot\frac{r}{n} + \frac{nt(nt-1)}{1.2}\cdot\frac{r^2}{n^2} + \&c.\right\} \text{ § 287. } e.$$

And when $n = \infty$,

$$A = p\left(1 + tr + \frac{t^2r^2}{1.2} + \frac{t^3r^3}{1.2.3} + \&c.\right)$$

$$= pe^{tr} = p(2.718\,281)^{tr}. \quad \text{§§ 330, 342.}$$

§ 273. No. of *permutations* of n things $= 1.2.3.4 \ldots n = [n]$.

§ 274. No. of *arrangements* of n things, taken p and p =

$$n(n-1) \ldots\ldots (n-p+1) = [n, n-p+1].$$

§ 275. No. of *combinations* of n things, taken p and p =

$$\frac{n(n-1)\ldots(n-p+1)}{1.2.3 \ldots p} = \frac{[n, n-p+1]}{[p]}.$$

§ 280. If $M + Nx + Px^2 + \&c. = 0$ for all values of x,

then $M = 0$; $N = 0$; &c.

§ 294. $(x+y)^n = x^n + \dfrac{n}{1}x^{n-1}y + \dfrac{n(n-1)}{1.2}x^{n-2}y^2 + \&c.$

§ 295. i.) $(x \pm y)^{\frac{p}{q}} = x^{\frac{p}{q}}\left(1 \pm \dfrac{p}{q}\dfrac{y}{x} + \dfrac{p(p-q)}{1.2\,q^2}\dfrac{y^2}{x^2} \pm \&c.\right)$

§ 300. $D_n = \pm a_1 \mp na_2 \pm \dfrac{n(n-1)}{1.2}a_3 \mp \&c.$;

taking the *upper* signs, if n is *even;* and the lower, if it is *odd.*

§ 301. $a_n = a_1 + (n-1)D_1 + \frac{(n-1)(n-2)}{1.2}D_2 + \&c.$

§ 304. $S = na_1 + \frac{n(n-1)}{1.2}D_1 + \frac{n(n-1)(n-2)}{1.2.3}D_2 + \&c.$

§ 307. $\frac{q}{m(m+p)} = \frac{1}{p}\left(\frac{q}{m} - \frac{q}{m+p}\right).$

§ 323. $\log y = M[y - 1 - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \&c.].$

§§ 327–8. $M = \frac{1}{a - 1 - \frac{1}{2}(a-1)^2 + \&c.} = \frac{1}{La} = le.$

§§ 329, 330. $M = .434\,294\,481.$ $e = 2.718\,281.$

§ 340. $a^x = 1 + La \,.\, x + \frac{(La)^2 x^2}{1.2} + \frac{(La)^3 x^3}{1.2.3} + \&c.$

§ 342. $e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$

§ 351. $x^n + A_1 x^{n-1} \,..\, + A_n = 0 = (x - a_1) \,..\, (x - a_n).$

§ 355. $A_1 = -(a_1 + a_2 \,\ldots\ldots\, + a_n);$

$A_2 = a_1 a_2 + a_1 a_3 + \&c.$ $A_n = \pm(a_1 a_2 \,..\, a_n).$

§§ 365–7. $X = x^n + A_1 x^{n-1} \,..\, + A_n = 0$, and $y = x - x'$.

$\therefore$ $Y = y^n + B_1 y^{n-1} \,\ldots\, + B_{n-1} y + B_n = 0;$

or $Y = y^n + \ldots\ldots \frac{f''(x')}{1.2}y^2 + \frac{f'(x')}{1}y + f(x') = 0.$

The parenthesis with the sign of equality, it will be observed, is sometimes used as an explanatory expression.

Thus (§ 18), "$10^{-1}(=\frac{1}{10})$" is used for "10^{-1} (i. e. $\frac{1}{10}$)."

ALGEBRA.

INTRODUCTION.

§ 1. ALGEBRA[a] is that branch of the science of number, which employs *general symbols*[b] *of quantity*[c].

a.) *Arithmetic*[d], in its largest sense, includes the whole science of number; but, in its popular use, the term is limited to that branch of the science, which employs symbols of known and particular numbers only; as 2, 3, 10, 12.

b.) *Algebra*, on the other hand, employs *general symbols* (for the most part, Italic letters of the alphabet), any one of which may represent any number whatever. Thus a represents, not some particular number, but simply a number.

NOTE. Such symbols are termed *algebraic* or *literal*[e], in distinction from those of common Arithmetic, which are termed *numerical*[f]. A quantity expressed algebraically is often called an *algebraic quantity* or *expression*.

c.) For convenience and perspicuity, certain classes of letters are usually appropriated to distinct uses. Thus, the first letters of the alphabet, as a, b, c, usually stand for *known* or *given* quantities, and the last, as x, y, z, for *un-*

(*a*) A word derived from the Arabic; the Arabs having been among the earliest cultivators of this science. (*b*) From the Greek σύμβολον, *token*, *sign*. (*c*) From the Latin quantus, *how much*. (*d*) Greek, ἀριθμός, *number*. (*e*) Latin, littera or litera, *a letter*. (*f*) Latin, numerus, *number*.

known or *required* quantities; while for exponents (§ 16), the letters near the middle of the alphabet, as m, n, p, are oftener used.

NOTE. A quantity is regarded as *known*, when it may be assumed at pleasure; as *unknown*, when it cannot be assumed, but must be found from its relation to the known quantities.

d.) A quantity is sometimes represented by the first letter, or by several letters of its name: thus interest is represented by i; sum, by S; difference, by D; time, by t; velocity, by v; radius, by r or R; sine, by *sin*; cosine, by *cos*; tangent, by *tan*[g]; &c.

e.) Different quantities of the same kind, or standing in the same circumstances, are sometimes represented by the same letter accented. Thus similar known quantities may be represented by a, a' (read *a prime*), a'' (*a second*), a''' (*a third*), &c.; similar unknown quantities by x, x', x'', &c. So, if the radius of one circle is represented by R, the radius of another may be represented by R', &c. A distinction is sometimes made, by using different forms of the same letter; as x, X; u, U; r, R.

SIGNS.

§ 2. In addition to the symbols of *quantity* above mentioned, Algebra, in common with other branches of mathematics, employs certain symbols of *operations* and *relations*, called SIGNS[h]. Thus, the sign of

a.) Equality, $=$, *equal to*; as 1 foot $=$ 12 inches; $a=b$.

b.) Inequality, 1. Superiority, $>$, *greater than*; as $10>7$.

2. Inferiority, $<$, *less than*; as $7<10$; $5a<6a$.

NOTE. The opening of the sign of inequality is always towards the greater quantity.

(*g*) Radius, sine, cosine, and tangent are the names of certain lines drawn in or about a circle, and express quantities of great importance, and of continual use in the higher applications of Algebra.

(*h*) Latin, signum, *mark*, *sign*.

c.) Addition, $+$, *plus*[i], or *together with;* as $6+4=10$; $x+a$.

d.) Subtraction, —, *minus*[j], or *less;* as $7-3=4$; $7a-3a$.

NOTE. The quantities, which are connected by the signs $+$ and —, are called terms[k].

e.) Multiplication, $\times$, or $.$, *into*, or *multiplied by;* as 4×5 or $4\,.\,5=20$; $4a\times5b=20a.b=20ab$.

NOTE. Between numbers and letters, and between letters themselves, the sign of multiplication is commonly omitted. Thus $3abc$ is the same as $3\times a\times b\times c$. Between numbers, on account of the local value of figures, the sign can never be omitted. Thus 35 is not the same as 3×5.

f.) Division, $\div$, *divided by;* as $8\div2=4$; $6a\div2=3a$.

NOTE. Division is more frequently denoted by writing the dividend above, and the divisor below a fractional line. Thus a divided by b is written $\frac{a}{b}$; $8\div2=\frac{8}{2}=4$.

g.) Inference, $\therefore$, *therefore*, as $a=5$, $\therefore 4a=20$.

h.) Union. The *parenthesis*, ( ), or *vinculum*[l], either horizontal, ——, or vertical, $|$, is used to show that several quantities, connected by the signs $+$ or —, are to be taken together, or subjected to the same operation. Thus $(3+4)\times2$, or $(3+4).2$, or $\overline{3+4}.2$, or $\left.\begin{array}{r}3\\+4\end{array}\right|2$, shows that 3 and 4 are to be added together, and their sum multiplied by 2. So $(a+b)(a-b)$; $6-(4-2)=6-2=4$. Without the parenthesis, the last expression would be $6-4-2=0$.

Other symbols will be introduced and explained, as they are needed.

§ 3. It should be remembered that these signs are abbreviations for words; that they are, in fact, words and phrases of the algebraic language.

(*i*) Lat. plus, *more.* (*j*) Lat. minus, *less.* (*k*) Gr. τέρμα, *bound, limit;* Lat. terminus, Fr. terme. (*l*) Lat., *tie, bond.*

a.) Translate the following expressions into common language.

1. $\frac{a+b}{2}+\frac{a-b}{2}=a.$

Ans. The quantity obtained by adding b to a and dividing the sum by 2, together with the quantity obtained by subtracting b from a and dividing the difference by 2, is equal to a.

Or, The half of a *plus* b, *plus* the half of a *minus* b, is equal to a.

2. $\frac{ab}{e}+\frac{xy}{a}-2a+ax\div y=rt\div v-\frac{xy}{b}.$

3. $(a+b)(c+x)=ac+bc+ax+bx.$

4. $R\times\sin(a+b)=\sin a\cos b+\cos a\sin b.$ See *b.* 4, below.

5. $ax+a'x'+a''x''+a'''x'''=(a+a'+a''+a''')X.$

6. $(100+40+4)12=144\,.\,\overline{10+2}=1728, <200\times 10.$

b.) Write, in algebraic language, the following sentences.

1. 10 added to 4, and the sum diminished by 8, is equal to 3 times 4 divided by 2.

Ans. $10+4-8=3\times 4\div 2.$

2. a multiplied by b, and the product divided by e, is equal to x multiplied by a, and the product diminished by b.

3. The difference between a multiplied by x, and h multiplied by y, is equal to m multiplied by e.

4. Radius into the sine of the sum of a and b is equal to the sine of a into the cosine of b, together with the product of the cosine of a into the sine of b. See *a.* 4, above.

5. The sum of a and b is greater than c, and c is greater than the difference of a and b.

The greater brevity and clearness of the algebraic language cannot fail to be observed.

POSITIVE AND NEGATIVE QUANTITIES.

§ 4. In finding the *aggregate* of any number of quantities, or *terms* (§2. *d.* N.), those, which tend to *increase* the amount, are called POSITIVE[m], and, as they must be *added*, are preceded by the sign $+$; those, which tend to *diminish* the result, are called NEGATIVE[n], and are preceded by the sign —, to show that they must be *subtracted.*

1. A has Bank Stock, to the amount of \$2000, Real Estate, \$5000, other property, \$1000; he owes to B \$500, and to C \$300. What is the net amount of his property?

Here the items of property tend to increase the amount, and are, therefore, *positive;* the debts diminish the amount, and are, therefore, *negative.* The former must, consequently, be preceded, or *affected* by, the sign $+$, and the latter, by the sign —. Hence, we shall have, for the true expression of the net value of the estate,

$$+2000+5000+1000-500-300=+\$7200.$$

a.) The character of every term as positive or negative, must, of course, be indicated in the expression. Quantities, however, are regarded as positive, unless the contrary is shown; hence, *if no sign* is prefixed to a term, the *sign* $+$ *is always understood.* Hence, when a positive term stands alone or at the beginning of a series of terms, its sign is usually omitted. Thus 5 is the same as $+5$;

so $4-3=+4-3$; $a=+a$; $a+b=+a+b$.

2. Let the items of property amount to \$10,000, the debts, to \$9000. What is the aggregate, or the net estate?

3. What is the aggregate, if the property be represented by a, and the debts by b?

(*m*) Lat. positivus, from pono, *to place*, as placing or giving value.
(*n*) Lat. negativus, from nego, *to deny*, as denying value.

4. Again, suppose a surveyor runs on one side of his field 20 rods east, and, on another, 15 rods west. What is his distance east of his starting point, i. e. his *departure*, as surveyors call it? *Ans.* $20-15=5$ rods.

or E. 20 rods, W. 15 rods = E. 5 rods.

The distance run east is positive, because it *increases* the distance east of the starting point; and the distance run west is negative, because it *diminishes* that distance.

b.) As each sign indicates simply the character *of the term before which it stands*, the *order* of the terms is obviously immaterial, provided each retains the proper sign before it. Thus $4-3$ is the same as $-3+4$. So,

$$10-8+6=10+6-8=6+10-8=-8+6+10.$$

5. How far will a surveyor be east of his starting point, if he runs 10 rods west, and 50 rods east?

Ans. $-10+50=50-10=40$ rods.

6. A owes \$5000, and holds property to the amount of \$20,000. What is his estate?

7. What, if he owes a dollars, and holds property to the amount of b dollars?

8. What, if he owes \$5000, and holds \$5000 worth of property?

9. What is his estate, if his property amounts to \$5000, and his debts, to \$6000? *Ans.* 5000—6000 = —\$1000.

or, property \$5000, debt \$6000 = debt \$1000.

In this instance, \$5000 of the debt can be paid, and there will remain \$1000 to be paid afterwards, i. e. to be *subtracted* from any property, which may be afterwards acquired.

10. A surveyor runs 20 rods east, and 30 rods west. What is his distance east of his starting point?

Ans. -10 rods.

or, E. 20 rods, W. 30 rods = W. 10 rods.

20 of the 30 rods run west can be subtracted from the 20 run east, and 10 remain to be subtracted. Thus, if he should afterwards run 15 rods east, his distance east of his first starting point would be $-10+15=5$ rods.

c.) If it had been proposed to find his *westerly* distance from the first point, the easterly distances would have been negative, and the westerly, positive.

In like manner, if we had proposed, in the examples above, to find the net *indebtedness*, we must have made debts positive, and property negative.

§ 5. Thus the contrary signs + and — show that the quantities, before which they are placed, *are in precisely opposite circumstances;* that is, that they produce opposite effects in respect to the aggregate result;—that, as in the case of the distance east and west, they are reckoned in opposite directions. In other words, the sign — is the algebraic expression for *contrariwise*, or, in reference to distances, *backwards.*

Thus, if distance north be positive, distance south is negative; if, for instance, north latitude have the sign +, south latitude must have the sign —. If distance upward be positive, distance downward is negative; if future time be positive, past time is negative; if velocity in one direction be positive, velocity in the opposite direction is negative; &c.

§ 6. A negative quantity is frequently said to be *less than zero.* This expression is most conveniently illustrated by examples 8 and 9, above. In example 8, the net estate is 0; in example 9, it is —$1000. But a man, whose property is as represented in example 9, is obviously poorer than he would be, if, as in example 8, he were worth simply nothing. He is *worth less than nothing.* It is not meant, that the thousand dollars to be subtracted, is less than zero; but, that it has less tendency to increase his estate, than zero would have; that is, it has a tendency actually to diminish his estate.

a. In like manner, if he had owed $2000, he would have been worth less than he is now, when he owes only $1000

Hence, we say, that $-2000 < -1000$. That is, the subtraction of 2000 leaves a smaller remainder than the subtraction of 1000. In other words, -2000 tends to increase the debt more, that is to increase the property less, than -1000, and is therefore said to be itself less.

So, in example 5, -10 gives a greater distance west, and therefore a less distance east, than -5 could have given; and either of them, a less distance east than 0. Hence,

$$0 > -1;\ -2 > -3;\ -5 < -4;\ +a > 0;\ -a < 0.$$

b. Again, if we begin with 3 and subtract 1, we diminish the amount; and we continue to diminish it, as long as we continue to subtract 1. Thus,

$$3-1=2;\ 2-1=1;\ 1-1=0;\ 0-1=-1;\ -1-1=-2.$$

Or, if, from the same quantity, we subtract continually greater and greater quantities, we shall obtain less and less remainders. Thus,

$$3-2=1;\ 3-3=0;\ 3-4=-1;\ 3-5=-2;$$

that is, the greater the quantity to be subtracted, the less the remainder.

§7. As a positive and negative quantity are reckoned in opposite directions, the *difference* between them is greater than either, and is equal to the sum of the units in both.

Or, as a negative quantity is *less* than zero, the difference between a positive and a negative quantity is greater than the difference between the positive quantity and zero; and greater by just so much as the negative quantity is less than zero; that is, by the number of units in the negative quantity.

1. A has \$5000, and B owes \$5000. What is the difference of their estates? i. e. by how much is A richer than B? *Ans.* $5000+5000=\$10,000$.

a.) If they should combine their estates, the *aggregate* value would be 0. The *difference* between them is clearly

\$10,000, the sum which B must obtain, in order to be as rich as A. This difference is expressed thus, 5000—(—5000). Hence,

$$5000-(-5000)=5000+5000; \text{ or } -(-5000)=+5000.$$

So $-(-a)=+a$. Hence,

b.) *The subtraction of a negative quantity has the same effect as the addition of an equal positive quantity.*

2. The latitude of New Orleans is 30° North; that of Buenos Ayres is 34° South. How many degrees is the one place North of the other? That is, what is the *difference* of their latitudes?

3. X has a dollars, and Y owes b dollars. What is the difference between their estates?

Ans. $a-(-b)=a+b$, as in example 1.

4. At sunrise on the 20th of February, the thermometer stood at 30° below zero; at sunrise on the 20th of March, it stood at 30° above zero. What is the difference in the temperatures?

5. The reading of the thermometer on one day is —10° (10° below 0); on another day, it is —20°. Which indicates the greater heat? How much? §6. a and b.

§ 8. The process of finding the *aggregate* of several quantities, regard being had to their character as positive or negative, is *algebraic addition;* the process of finding the *difference* between quantities so considered is *algebraic subtraction. Arithmetical* addition and subtraction, on the other hand, relate to numbers regarded simply as such, without distinguishing them as positive and negative.

(a) The *algebraic sum* may be less than the *algebraic difference* (§7. a); and (b) the *algebraic sum* may be equal to the *arithmetical difference* (§4); or (c) the *algebraic difference*, to the *arithmetical sum*.

FACTORS AND POWERS.

§ 9. *Quantities multiplied together* are called, as in Arithmetic, FACTORS[o] in respect to the product[p], and are also called COEFFICIENTS[q] in respect to each other.

Thus, in the expressions $3a$, $2a$, ba, bca, and $\frac{1}{2}a$, 3, 2, b, bc and $\frac{1}{2}$ are coefficients of a. In $3xy$, 3 is the coefficient of xy; $3x$, of y; and $3y$, of x.

a.) The coefficient shows, how many times the quantity multiplied is taken as a *term* (§2. *d.* N). If the coefficient is *positive*, it shows how many times the quantity is *added*; if *negative*, how many times it is *subtracted* (§4). Thus,

$$3a = a + a + a;\ 2x = x + x.$$

$$-3 \times +a = -a - a - a = 3 \times -a = -3a.$$

$$\text{So } -a \times +b = a \times -b = -ab.$$

$$-2 \times -a = -(-a) - (-a) = a + a\ (\S\, 7.\ a,\ b) = 2a.$$

NOTE. In the last example, $-a$ is to be *subtracted* twice; and subtracting $-a$ twice has the same effect as adding $+a$ twice (§7 *b*).

Hence, if two factors multiplied together are *both positive* or *both negative, the product is positive*; if *one is positive and the other negative, the product is negative.* Or, more briefly,

LIKE *signs give* $+$, UNLIKE, $-$.

1. What is the product of $2a$ and $-b$? of $-2ab$ and $-c$?

2. $3a \times -xy =$ what? $-3a \times -xy$? $-3a \times -xy$? -2×-3?

b.) A *letter*, or combination of letters, used as a coefficient, is called a *literal* coefficient; a *number*, so employed, is called a *numerical* coefficient. Coefficients are also distinguished as *integral*[r] or *fractional*, &c.

(*o*) Lat., *maker, producer.* (*p*) L. productus, *produced,* i e. by the multiplication. (*q*) Lat. coefficio. *to aid in forming*, a co-factor. (*r*) Lat. integer, *whole;* numbers are called *integral* or *whole,*

When no numerical coefficient is expressed, 1 is always implied. Thus a is the same as $1a$; $x=1x$; $ab=1ab$.

1. In $7abcx$, what is the coefficient of x? of cx? of bcx?

2. In xyz, what is the coefficient of x? of y? of xyz?

3. In $(a+b)(a-b)c$, what is the coefficient of c? of $a+b$? of $a-b$?

4. In $-5abx$, what is the coefficient of x? of $5x$? of $-5x$? $-5a$? $-ab$? abx? $-abx$?

§ 10. The *combining* of factors into a product is the work of *multiplication;* the *separation* of a given factor from a *given product* is the work of *division.*

Thus, by multiplication, we combine the factors, 3 and 4, into a product 12: by division, we separate the given factor 3, from the given product 12, and find the other factor 4.

a.) The *given product* is called, in reference to division, the *dividend*[s]; the *given factor* the *divisor*[t]; and the *required factor*, the *quotient*[u].

b.) The divisor and quotient are the factors of the dividend. They are, therefore, *coefficients* of each other. If then the letters of the divisor be found in the dividend, we have only to *suppress* or *cancel* them, and the *remaining factors constitute the quotient* (division of the numerical coefficients being performed as in Arithmetic).

Thus $ab \div b=a$; $abx \div ab=x$; $7abcxy \div 7ac=bxy$.

1. $2abx \div b=$ what? $16abcxyz \div 8abz$?

2. $3.4.5.6 \div 3.6=$ what? $1.2.3.4.5.6 \div 6.5.4$?

3. $a \div a=$ what? $ab \div ab$? $1.2.3 \div 1.2.3$?

NOTE. When the divisor is equal to the dividend, the quotient is obviously unity.

---

in distinction from *fractional* (Lat. frango, *to break*), or *broken* numbers. (*s*) Lat. Dividendus, *to be divided.* (*t*) Lat. divisor, *a divider* from divido, *to divide*, or *separate.* (*u*) Lat. quoties, or quotiens, *how many times*, as it shows how many times the divisor is contained in the dividend.

c.) If the divisor contain factors, which are not found in the dividend, we may cancel the common factors, and *express* the division by the remaining factors of the divisor in the usual form (§2. *f* and N).

Thus $2\cancel{a}bc \div \cancel{a}x = 2bc \div x$, or $\frac{2bc}{x}$.

1. $\frac{7abxy}{bcxr} =$ what? $\frac{abx}{bc}$? $\frac{2bcx}{3bc}$? $\frac{\sin a \cos b}{\cos a \cos b}$?

2. $\frac{1.2.3.4}{4.5.} =$ what? $\frac{20}{15}$? $\frac{5.4.3.2.1}{1.2.3}$?

NOTE. This, it will be observed, is equivalent to the process of reducing a fraction to its lowest terms. This process may be applied in all cases. Whenever all the factors can be cancelled out of either the divisor or the dividend, unity will be found in their place. If this happen to the divisor, the quotient will be found in the usual form as above (*b*); if to the dividend, unity will stand above the line, or in the place of the dividend, and the remaining factors of the divisor will stand below the line, or after the sign; if to both divisor and dividend, the result will be $1 \div 1 = 1$.

d.) If the dividend is *positive*, its factors (the divisor and quotient) must have *like signs* (both positive, or both negative); and if the dividend is *negative*, its factors must have *unlike signs* (one positive, and the other negative) (See § 9. *a*). Therefore,

If the *dividend is positive*, a *positive divisor* gives a *positive quotient;* a *negative divisor*, a *negative quotient;* if the *dividend is negative*, a *positive divisor* gives a *negative quotient;* a *negative divisor*, a *positive quotient*. Hence, as in multiplication,

LIKE *signs* give +, UNLIKE, —.

Thus, $\frac{+ab}{+a} = +b$, for $(+a)(+b) = +ab$; $\frac{+12}{+3} = +4$.

$\frac{-ab}{-a} = +b$, for $(-a)(+b) = -ab$; $\frac{-12}{-3} = 4$.

$\frac{+ab}{-a} = -b$, for $(-a)(-b) = +ab$; $\frac{+12}{-3} = -4$.

$$\frac{-ab}{+a} = -b, \text{ for } (+a)(-b) = -ab; \quad \frac{-12}{+3} = -4.$$

1. $-2ab \div -2a =$ what? $-2ab \div 2a$? $2ax \div -a$?
2. $-10x \div -10 =$ what? $60 \div -10$? $-60 \div -10$?

§ 11. When a factor occurs more than once in a product, it is usually *written* but once, and *the number of times it is employed*, is denoted by a number or letter placed over it at the right, called an EXPONENT[v], or *index*[w].

Thus, instead of aa, aaa, $bbbbbb$, we write a^2, a^3, b^6; instead of 2.2, 2.2.2.2, 3.3.3.3.3.3, we write 2^2, 2^4, 3^6, the exponent, in every case, showing how many times the quantity over which it is placed is taken as a factor; in other words, how many *equal factors* the product contains. Thus, in the expression, $(a+b)^3$, the exponent ^{3} shows that $a+b$ is taken three times as a factor, or that the product consists of three factors each equal to $a+b$. So, the product $a^2b^3x^5$ contains two factors equal to a, three equal to b, and five equal to x.

1. Write 2.2.3.2.3.2 with exponents. *Ans.* $2^4.3^2$.
2. Write *aabcabac* with exponents.
3. Write $2^3.10^3.3^4.5^4$ without exponents.
4. Write $a^4b^3c^2x^5y^6$ without exponents.

NOTE 1. These expressions may be read thus; a^2, a taken twice as a factor; b^3, b taken three times as a factor; &c. Also, a^1, (§ 11. a), a^0 (§ 13), a taken once, a taken no times as a factor; $a^{\frac{1}{2}}$ (§ 12) a taken half a time as a factor; a^{-2} (§ 14), a taken *minus* twice as a factor; &c. Or, if the teacher prefer, the student may examine § 22 and a under it, and use the expressions given there.

NOTE 2. A *negative* quantity may obviously occur more than once as a factor; as $(-a)(-a) = (-a)^2$; $(-b)(-b)(-b) = (-b)^3$. In such cases, if the number of factors be *even*, the product will be positive; for, if they be combined two and two, the product of each

(v) Lat. exponens, *setting forth, showing*. (w) Lat. *indicator, mark*.

pair will be positive (§ 9. *a*); and the product of these positive products will, of course, be positive. If the number of factors be *odd*, the greatest even number will give a positive product, and this, multiplied by the remaining negative factor, will give a negative product (§ 9. *a*). Hence,

If the number of negative factors be *even*, the product will be *positive;* if *odd*, *negative*. Thus,

$$(-a)^2=+a^2;\ (-a)^3=-a^3;$$
$$(-x)^4=+x^4;\ (-x)^5=-x^5.$$

a.) When a quantity is taken as a factor only once, the fact may be shown by the exponent ^{1}; but in this case, the exponent is usually not written; and whenever *no exponent is written*, 1 *is always implied.* Thus a is the same as a^1; $ax=a^1x^1$; $ax^2=a^1x^2$.

§ 12. *b.*) The fraction $\frac{1}{2}$ shows that the unit is separated into two equal parts, and that only one of them is taken. So the exponent $\frac{1}{2}$, in the expression $a^{\frac{1}{2}}$, shows that a is *separated into two equal factors*, and that only one of them is employed; in other words, that a is introduced as a factor, *half a time.* If this *half-factor* were introduced *two*, *three*, or *four* times, we should have $a^{\frac{2}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{4}{2}}$. Thus,

$$a^{\frac{4}{2}}=a^{\frac{1}{2}}.a^{\frac{1}{2}}.a^{\frac{1}{2}}.a^{\frac{1}{2}}=(a^{\frac{1}{2}})^4.$$

If a were separated into *three*, *four*, or n equal factors, and *one* only employed, we should write $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{n}}$; if *two* were employed, $a^{\frac{2}{3}}$, $a^{\frac{2}{4}}$, $a^{\frac{2}{n}}$; &c. Hence,

The *denominator* of a fractional exponent shows, *into how many equal factors* the quantity under the exponent is *separated;* and the *numerator* shows, how many of these factors are employed. Thus,

$$9^{\frac{1}{2}}=3;\ 9^{\frac{2}{2}}=9^{\frac{1}{2}}.9^{\frac{1}{2}}=3.3=9;\ 9^{\frac{3}{2}}=3.3.3=27.$$
$$8^{\frac{1}{3}}=2;\ 8^{\frac{2}{3}}=2.2=4;\ 8^{\frac{4}{3}}=2.2.2.2=16.$$

1. What is the meaning of $(aa)^{\frac{3}{2}}$? of $R^{\frac{3}{2}}$? of $a^{\frac{6}{3}}$?

2. $(R^2)^{\frac{5}{2}}=$ what? $16^{\frac{3}{2}}$? $27^{\frac{2}{3}}$? $25^{\frac{1}{2}}$? $36^{\frac{3}{2}}$? $49^{\frac{1}{2}}$?

c.) Otherwise, as $\frac{4}{2} = \frac{1}{2}$ of 4, $a^{\frac{4}{2}}$ indicates, that *one half of four* factors each equal to a are introduced; or that a had been introduced *four* times as a factor, and the product, so formed, had been afterwards separated into *two* equal factors, of which only one was actually employed. Thus,

$$a^{\frac{4}{2}} = a^{4 \times \frac{1}{2}} = (a^4)^{\frac{1}{2}} = (aaaa)^{\frac{1}{2}} = aa = a^2.$$

Hence, again,

The *numerator* of a fractional exponent shows, how many times the quantity under the exponent has been employed as a factor; and the *denominator* shows, into *how many equal factors the product so formed has been separated.*

Thus $a^{\frac{1}{3}}$, $a^{\frac{2}{3}}$, $a^{\frac{3}{4}}$, $a^{\frac{m}{n}}$ indicate, that a, a^2, a^3, a^m have been separated, the first two into 3, the third into 4, and the fourth into n equal factors, of which only one is employed. Or that a is employed as a factor $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{m}{n}$ of a time.

Thus, $9^{\frac{2}{2}} = (9^2)^{\frac{1}{2}} = (9.9)^{\frac{1}{2}} = 9$; $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (8.8)^{\frac{1}{3}} = 64^{\frac{1}{3}} = (4.4.4)^{\frac{1}{3}} = 4.$

$(R^2)^{\frac{3}{2}} = (R^2.R^2.R^2)^{\frac{1}{2}} = (R.R.R.R.R.R.)^{\frac{1}{2}} = (R^6)^{\frac{1}{2}} = R^3.$

1. What is the meaning of $(aa)^{\frac{3}{2}}$? of $2^{\frac{4}{2}}$? of $3^{\frac{4}{2}}$? of $3^{\frac{3}{2}}$? of $x^{\frac{2}{3}}$?

2. $(R^2)^{\frac{5}{2}} =$ what? $16^{\frac{2}{4}}$? $27^{\frac{4}{3}}$? $5^{\frac{4}{2}}$? $2^{\frac{6}{2}}$? $(x^3)^{\frac{2}{3}}$?

§ 13. *d.*) Any quantity, which is not found as a factor in a product, may be introduced with *zero* for an exponent. For this exponent will show, that the quantity, though written, still is not employed, or is employed *no times*, as a factor; and, of course, the value of the expression is the same as if the quantity were not written. Thus a^0bx^2 is the same as bx^2; $ax^0 = a$; but $a \times 1 = a$; that is, $a \times x^0 = a \times 1$; $\therefore x^0 = 1$. Hence,

Corollary I. *Any quantity with zero for its exponent is equal to unity.*

NOTE. A COROLLARY[x] is an *inference from a preceding principle.*

§ 14. *e.*) When a factor is introduced *less than no times* (§ 6.), i. e. when instead of being *introduced*, it is *taken out*, the fact will be properly indicated by a *negative* exponent (§§ 4, 5). But a factor is *taken out* by *division* (§ 10). Consequently, a negative exponent shows, that the quantity under it is to be employed as a *divisor*, as many times (§ 11), or parts of a time (§ 12. *b*, *c*), as there are units or parts of a unit in the exponent. Thus, in the expression $a^{-1}x$, a, instead of being *multiplied into*, is to be *divided out of* x, and the expression is therefore equivalent to $\frac{x}{a}$.

Also, in the expression $a^{-\frac{5}{2}}x$, the *negative fractional* exponent $^{-\frac{5}{2}}$ indicates, that a is separated into two equal factors, and that one of these *half-factors* (§ 12. *b*) is *taken out* five times by *division*; i. e. that the whole factor a is taken out *five halves of a time*. This is evidently the same thing as saying, that it is *introduced minus* five halves of a time. In other words $a^{-\frac{5}{2}}$ indicates, that a product, containing a five times as a factor, is separated into two equal factors, and that one of these two factors is to be taken out by division. The expression is, therefore equivalent to $\frac{x}{a^{\frac{5}{2}}}$.

So,

$$a^{-2}x = \frac{x}{a^2};\ 10^{-1}.12 = \frac{12}{10};\ 2^{-3}.5^2 = \frac{5^2}{2^3} = \frac{25}{8};$$

$$9^{-\frac{1}{2}}.6 = \frac{6}{9^{\frac{1}{2}}} = \frac{6}{3} = 2. \quad \text{See §§ 17, 19.}$$

1. $2^{-1}.3 =$ what? $3^{-1}.2$? $10^{-2}.30$? $15^{-1}.30$?

(x) Lat. corollarium, *something given over and above*, from corolla, *a wreath*, a common present or mark of honor.

2. $a^{-3}b =$ what? $a^0b^{-1}x$? $b^{-\frac{1}{2}}x$? $a^{\frac{1}{3}}c^0x^{-1}$? $a^{-m}x^{n}$?

§ 15. *f.*) If a quantity be found any number of times in a multiplier and multiplicand, it will be found in the product as many times as in both the factors. For

$a^2b^3 \times a^4b = aabbb \times aaaab = aaaaaabbbb$ (§2. *e.* N) $= a^6b^4$. Hence,

Cor. II. *The exponent of any quantity in a product will be equal to the sum of its exponents in the factors.*

1. $a^2b^2 \times ab =$ what? $ax^2 \times a^2x$? $a^0bc^2 \times a^2bc^3$? $a^3 x^0 \times a^3y$?

2. $2^3.3^3 \times 2.3^2 =$ what?

Ans. $2^4.3^5 = 16 \times 243 = 3888$.

3. $2^2.3^4 \times 2^0.3 =$ what? $5^3.2 \times 5^0 2$? $10^2 \times 10^3$?

4. $100^{\frac{1}{2}} \times 100^{\frac{3}{2}} =$ what?

Ans. $10.10^3 = 10^4 = 10{,}000$.

5. $100^{\frac{1}{2}} \times 100 =$ what? $25^{\frac{1}{2}}.25^{\frac{3}{2}}$? $27^{\frac{1}{3}}.27^{\frac{2}{3}}$? $16^{\frac{1}{4}}.16^{\frac{5}{4}}$?

6. $a^{\frac{1}{2}} \times a^{\frac{1}{2}} =$ what? $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$? $16^{\frac{1}{2}}.16^{\frac{1}{2}}$? $10^{\frac{1}{2}}.10^{\frac{3}{2}}$?

§ 16. *g.*) It is also evident, that the exponent of a quantity in one of the factors must be equal to the exponent of that quantity in the product, *minus* its exponent in the other factor. Hence,

Cor. III. *The exponent of any quantity in a quotient is equal to the exponent of that quantity in the dividend,* minus *its exponent in the divisor.*

Thus $\dfrac{a^5}{a^3} = \dfrac{aaaaa}{aaa} = aa = a^2$; $\dfrac{10^5}{10^3} = 10^2$; $\dfrac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}} = a^{\frac{4}{2}} = (aaaa)^{\frac{1}{2}} = a^2$.

1. $\dfrac{x^3}{x^2} =$ what? $\dfrac{a^7b^3}{a^4b}$? $\dfrac{a^2b^2}{ab}$? $\dfrac{a^3b^2}{ab^2}$? $\dfrac{ax^5}{a^0x}$?

2. $\dfrac{x^3}{x^3} =$ what? *Ans.* $x^{3-3} = x^0 = 1$ (Cor. I).

3. $a^{\frac{5}{2}} \div a^{\frac{3}{2}} =$ what? $10^2 \div 10^{\frac{3}{2}}$? $a \div a^{\frac{3}{2}}$? $a^{\frac{1}{2}} \div a^2$?

4. $\frac{x^3}{x^4} =$ what? *Ans.* $x^{3-4} = x^{-1}$, or $\frac{x^3}{x^4} = \frac{1}{x}$. Hence,

§ 17. *h.*) We have $x^{-1} = \frac{1}{x}$. See § 14.

In like manner, $\frac{x^3}{x^5}, \frac{x^3}{x^6}, \frac{x^3}{x^7}, \frac{x^3}{x^{n+3}}$, give x^{-2}, x^{-3}, x^{-4}, $x^{-n} = \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \frac{1}{x^n}$, respectively. Hence,

Cor. IV. *A quantity with a negative exponent is equal to unity divided by the same quantity with an equal positive exponent.*

§ 18. *i.*) The quotient obtained by *dividing unity by any quantity* is called the RECIPROCAL[y] of that quantity. Thus $\frac{1}{x}, \frac{1}{x^2}, 1 \div a, \frac{1}{10}, \frac{1}{10^2}, 1 \div a^{\frac{1}{2}}$, or the equivalent expressions $x^{-1}, x^{-2}, a^{-1}, 10^{-1}, 10^{-2}, a^{-\frac{1}{2}}$, are the reciprocals of x, x^2, a, 10, 10^2 and $a^{\frac{1}{2}}$ respectively. Also the reciprocal of $10^{-1} (= \frac{1}{10})$ is $\frac{1}{10^{-1}} = 1 \div \frac{1}{10} = 10$;* the reciprocal of $a^{-2} \left(= \frac{1}{a^2}\right)$ is $\frac{1}{a^{-2}} = 1 \div \frac{1}{a^2} = a^2$.* Hence,

To express the reciprocal of any quantity, we have only to *change the sign of its exponent.*

Write the answers to the following questions both by means of exponents with their signs changed, and under the fractional form.

What is the reciprocal of 2? of 3? of 10? of $\frac{1}{2}$? of $\frac{1}{3}$? of $\frac{1}{10} (= 10^{-1})$? of $\frac{1}{10^2}$? of .01? of x^2? of x^{-3}? of 5^{-3}? of $9^{\frac{1}{2}}$? of $8^{\frac{1}{3}}$? of $25^{-\frac{1}{2}}$?

(*y*) Lat. reciprocus, *returning upon itself, mutual.*

* NOTE. This is evidently true; for, if a unit be divided into 10 equal parts, one of them will be contained in any quantity 10 times as often as the whole unit is contained in the same quantity; and, if the unit be divided into a^2 equal parts, one of these parts will be contained a^2 times as often as the whole unit.

k.) Positive and negative *exponents* have the same relation of contrariety or *oppositeness* as other positive and negative quantities (§5). Thus, an exponent shows, how many times or parts of a time a quantity is *introduced as a factor*. The opposite to *introducing* a factor is *taking it out*. When therefore a quantity is said to be introduced *minus* three, or *minus* n times, as a factor, it is the same thing as saying that it must be *taken out* three, or n times (§ 14). Thus, in example fourth (§ 16), x can be taken out three times, and the fact, that it is to be taken out once more, is indicated by the negative exponent -1 (§ 4. *c*). It is to be so taken out, whenever, in subsequent multiplication, x shall be introduced.

If the operation be represented and performed in the fractional form, we have $\frac{x^3}{x^4}=\frac{1}{x}$; that is, three of the four factors of the divisor are cancelled out of the dividend, and one remains to be taken out, whenever x shall be introduced into the dividend.

§ 19. *l.*) As the factors under the negative exponent diminish the whole number of factors in the product, therefore,

(1.) In combining the exponents of a letter in the factors, to find its exponent in the product, the negative *exponents* must be treated precisely as negative *terms* in making up an aggregate (§4). Thus $a^3\times a^{-1}=a^2$; $x^5\times x^{-4}=x^1=x$.

1. $x^8\times x^{-6}=$ what? $x^{13}\times x^{-4}$? $a^2b^2\times a^{-1}b$?

2. $x^2\times x^{-5}=$ what? *Ans.* $x^{2-5}=x^{-3}$.

But
$$\frac{x^2}{x^5}=x^{2-5}=x^{-3}.$$

$$\therefore\quad x^2\times x^{-5}=x^2\div x^5.$$

(2.) In like manner, if negative exponents be found in a divisor or dividend, they must be treated like negative terms in finding a difference (§7). Thus,

$$a^2 \div a^{-3} = a^{2-(-3)} = a^5. \quad \text{See § 7. } a, b.$$

But $$a^2 \times a^3 = a^5.$$

$\therefore$ $$a^2 \div a^{-3} = a^2 \times a^3.$$

Hence (1, 2),

Cor. IV. *To multiply or divide by a quantity with a negative exponent, is the same as to divide or multiply by the quantity with an equal positive exponent.*

Or, more generally,

To MULTIPLY *or* DIVIDE *by any quantity is the same as to* DIVIDE *or* MULTIPLY *by its reciprocal.*

Thus $ab^2cx^2y \div abc = ab^2cx^2y \times a^{-1}b^{-1}c^{-1} = bx^2y.$

$$2^2.3 \div 2.3 = 2^2.3 \times 2^{-1}.3^{-1} = 2.$$

1. $abc \div abc =$ what? $abc \times a^{-1}b^{-1}c^{-1}$? $a^2x \times a^{-1}x$? $a^2x \div ax^{-1}$?

2. $2^2.3^2.4^2 \div 2.3.4 =$ what? $2^2.3^2.4^2 \div 2^{-1}.3^{-1}.4^{-1}$? $2.3.10 \div 2.3$? $2.3.10 \div 2^{-1}.3^{-1}$?

3. $a^{\frac{1}{2}}.a^{-\frac{1}{2}} =$ what? $a^{-\frac{1}{2}}.a^{-\frac{1}{2}}$? $10^{-\frac{1}{2}}.10^{-\frac{3}{2}}$? $x.x^{-\frac{1}{2}}$?

4. $a^{\frac{1}{2}} \div a^{-\frac{1}{2}} =$ what? $x \div x^{-\frac{1}{2}}$? $a^{-\frac{1}{2}} \div a^{-\frac{1}{2}}$? $10 \div 10^{-\frac{1}{2}}$?

§ 20. *m.*) When a quantity is taken as a factor any number of times, and the *product* so formed is again taken as a factor any number of times, the first quantity will evidently be employed a number of times equal to the *product of the exponents.* (See § 12. Examples.) Thus,

$$(a^3)^3 = a^3.a^3.a^3 = a^9; \; (a^{\frac{1}{4}})^3 = a^{\frac{1}{4}}.a^{\frac{1}{4}}.a^{\frac{1}{4}} = a^{\frac{3}{4}};$$

1. $(a^{-4})^3 =$ what? $(2^3)^2$? $(10^2)^{\frac{1}{2}}$? $(x^4)^{\frac{1}{8}}$? $(a^8)^{-\frac{1}{8}}$? $(a^m)^n$? $(2^5)^3$?

n.) Thus we see that the exponent may be either *integral* or *fractional, positive* or *negative,* and it may be either *known* or *unknown.*

§ 21. *o.*) The analogy, as well as the difference, between the *coefficient* (§ 9. *a*) and *exponent,* is very obvious. Both relate to the introduction of *equal* quantities; the *coeffi-*

cient, of equal *terms* (§ 2. *d*. N); the *exponent*, of equal *factors* (§ 9). If *positive*, they *affirm* the introduction of the quantities; the *coefficient*, by *addition* (§ 9. *a*); the *exponent*, by *multiplication* (§ 11). If *negative*, they *deny* the introduction, i. e. they affirm the *removal* or *taking out* of the quantities; the *coefficient*, by *subtraction* (§ 9. *a*); the exponent by *division* (§ 14). If *fractional*, they show the introduction or removal, by addition or subtraction, or by multiplication or division, as the case may be; the *coefficient* of equal *fractional parts* (§ 9. *b*); the exponent, of *equal components*[z] (§ 12) of the quantity.

That is, they show, how many times or parts of a time, a quantity is introduced or taken out; the *coefficient*, as a *term*; the *exponent*, as a *factor*. In other words, they show the introduction, *positively* or *negatively* (§ 4), of a term or factor, so many times as there are units in the coefficient or exponent.

Thus $+2\times4=4+4=+8$; $4^{+2}=4\times4=16$.

$$-2\times4=-4-4=-8;\quad 4^{-2}=\frac{1}{4^2}=\frac{1}{4.4}=\frac{1}{16}.$$

$$+\tfrac{1}{2}.16=\tfrac{1}{2}(8+8)=8;\quad 16^{+\frac{1}{2}}=(4.4)^{\frac{1}{2}}=4.$$

$$-\tfrac{1}{2}.16=\tfrac{1}{2}(-8-8)=-8;\quad 16^{-\frac{1}{2}}=\frac{1}{16^{\frac{1}{2}}}=\frac{1}{(4\times4)^{\frac{1}{2}}}=\frac{1}{4}.$$

So, $x+0\times a=x+0=x$; $x\times a^0=x\times1=x$.

1. Write $abbbccxxxx$ with exponents.
   Ans. $a^1b^3c^2x^4$.
2. Write in like manner, $aayy$, $bxbx$, $(a+b)(a+b)$.
3. Write $a^2b^3x^1y^0$ without exponents. *Ans.* $aabbbx$.
4. Write in like manner, a^4x^5, $(a+b)^2$, $(a-b)^3$.
5. Write with exponents $2\times2\times3\times2\times3\times4$.
   Ans. $2^3.3^2.4^1$.
6. Write with exponents $2\times2\times2\times3\times3\times2\times3\times4\times4$.
7. $4^3=$ what? *Ans.* $4\times4\times4=64$.

(z) Lat. compono, *to compose*; factors, which, multiplied together, produce a quantity, are called its *components*.

8. 4^4 = what? 5^3? 7^2? 10^4? 10^5? 21^2?

9. What is the difference between 10×4 and 10^4?

10. Show the difference between $3a$ and a^3?

Ans. $3a = a + a + a$; $a^3 = a \times a \times a$.

11. What is the difference between a^0 and $a \times 0$? between 10^0 and 10×0? between 1^0 and 1×0? between 0^0, 1^0 and 10^0? between 1^0, 1^1 and 1^2?

12. Show the difference between $\frac{1}{2}a$ and $a^{\frac{1}{2}}$.

Ans. $\frac{1}{2}a + \frac{1}{2}a = a$, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$.

13. What is the difference between $100 \times \frac{1}{2}$ and $100^{\frac{1}{2}}$? between $\frac{1}{2}$ of 9 and $9^{\frac{1}{2}}$? between $\frac{1}{3}.27$ and $27^{\frac{1}{3}}$? between $\frac{3}{2}.16$ and $16^{\frac{3}{2}}$?

14. What is the difference between -3^2, 3^{-2}, $3-2$ and $3(-2)$?

Ans. $-3^2 = -9$; $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$; $3-2 = 1$; $3(-2) = -6$.

15. Write in like manner 6, 8, 10 and 15, and interpret the expressions.

16. What is the difference between $9^{\frac{1}{2}}$ and $9^{-\frac{1}{2}}$? $8^{\frac{2}{3}}$ and $8^{-\frac{2}{3}}$?

17. What is the reciprocal of 10? of 10^2? of 100? of -10? of 1? of a? of $\frac{1}{10}$? of $\frac{1}{a}$? of a^{-1}? of $a^{\frac{1}{2}}$? of a^{-n}? of $100^{\frac{1}{2}}$? of $27^{-\frac{1}{3}}$? of $8^{\frac{4}{3}}$?

18. $a^2 \div a$ = what? $a^2 \div a^0$? $a^2 \div a^2$? $a^2 \div a^3$?

19. $a^3 \div a$ = what? $a^3 \div a^0$? $a^3 \div a^{-1}$? $a^3 \div a^{-2}$?

20. Substitute 10 for a in the last two examples.

22. If a is employed m times, and b, n times, what is the expression for their product?

* § 22. *Any quantity with an exponent*, is called a POWER of the quantity under the exponent.

NOTE. The quantity under the exponent is called the BASE[a] of the power.

a.) A power is *designated* by its exponent. Thus,

x^{-2} is read x *minus* second power; x^{-1}, x *minus* first power.

x^0 " x zero power; x^1, x first power.

x^2 " x second power or *square*[b]; x^3, x third power or *cube*[c].

$x^{\frac{1}{2}}$ " x one half power; $x^{\frac{2}{3}}$, x two thirds power.

$x^{-\frac{1}{2}}$ " x *minus* one half power, &c.

b.) It will be observed, that the term power, as used here, has a wider signification than is attached to it in Arithmetic. In Arithmetic, the term is applied only to a product of equal factors. As here defined, it includes a *single factor* (§ 11. *a*), *unity* (equal to the zero power (§ 13) of a factor), and all *products* and *quotients* formed by multiplying and dividing (§ 14, 17) unity, any number of times, by the factor, or by any of its equal components (§ 12, 14).

c.) We have therefore several classes of powers, distinguished by the characters of their exponents. Thus, there are

(1.) Powers with *positive integral* exponents (§ 11), the same as ordinary arithmetical powers;

(2.) Powers with *positive fractional* exponents (§ 12), consisting of equal *components* and their combinations;

(3.) Powers with *negative integral* exponents (§ 14), the reciprocals of the first class; and

(4.) Powers with *negative fractional* exponents (§ 14), the reciprocals of the second class.

d.) Powers of these several classes are sometimes called *positive*, *negative*, &c., *powers;* meaning, not that they are positive or negative, integral or fractional *quantities*, but

---

(*a*) Gr. βάσις, *foundation*. (*b*) Lat. quadra, Fr. quarré; because the second power of a factor represents the surface of a square, whese side is represented by the factor (Geom. §§ 124, 171, 177). (*c*) Gr. κύβος; because the third power of a factor represents the solid content of a cube, whose edge is represented by the factor.

that they have such *exponents.* So a power, whose exponent is an even number, is frequently called an *even* power; one whose exponent is an odd number, an *odd* power.

§ 23. *One of the equal factors* (§ 12) of a quantity is called its ROOT.

a.) A *root* is called the *second* or *square*, the *third* or *cube*, the *fourth*, the nth, according as it is one of *two*, *three*, *four*, or n equal factors, which produce the given quantity; i. e. into which the given quantity is separated.

Thus 2 is the *third* or *cube* root of 8, because it is one of three equal factors, which produce 8. So a is the third root of a^3, the fourth root of a^4; $a^{\frac{3}{2}}$ is the second or square root of a^3, because it is one of the two equal factors, into which a^3 may be separated (§ 12. *c*).

b.) A root of any quantity is properly expressed by writing the quantity under a fractional exponent, whose numerator is unity, and whose denominator is equal to the number of the root (§ 12. *c*). For this denotes, that the quantity under the fractional exponent is separated into so many equal factors, as there are units in the denominator, and that only one of them is taken. Thus,

The second or square root of a is $a^{\frac{1}{2}}$; that of a^2 is $(a^2)^{\frac{1}{2}} = a^{2\times\frac{1}{2}}$ (§ 20) $= a$; the third or cube root of a is $a^{\frac{1}{3}}$; that of a^2 is $(a^2)^{\frac{1}{3}} = a^{\frac{2}{3}}$.

c.) The principle of § 12. *b*, *c* may, therefore, be expressed as follows:

A fractional exponent shows, either that the root of the base, denoted by the denominator, is raised the power denoted by the numerator (§ 12. *b*); or, that the base being raised to the power denoted by the numerator, the root denoted by the denominator is taken (§ 12. *c*). Thus, $a^{\frac{5}{2}}$ expresses the fifth power of the second or square root of a; or the square root of the fifth power of a; so $8^{\frac{2}{3}}$ is equal

to the square of the cube root of 8; or to the cube root of the square of 8.

d.) A root is also frequently indicated by the *radical*[d] *sign*, $\surd$[e], placed before the quantity, with a number over the sign, to show the number of the root. In expressing the second or square root, however, the number is more frequently omitted; and, accordingly, wherever the sign stands without a number over it, it must always be understood to denote the square root. Thus,

$\sqrt{4}=4^{\frac{1}{2}}=$ the second or square root of 4.

$\sqrt[3]{8}=8^{\frac{1}{3}}=$ " third or cube " " 8.

$\sqrt[5]{a}=a^{\frac{1}{5}}=$ " fifth " " a.

NOTE. Either of these forms of expressing the root, may be used at pleasure, and both should be made familiar. The fractional exponent is, however, generally, more *convenient* than the radical sign; and is, besides, to be preferred because it exhibits roots as a class of *powers*, and enables us to refer the operations upon roots to the general principles, which govern the operations upon powers. Quantities written under a radical sign are frequently called *radical quantities*.

e.) As the product of an *odd* number of positive factors is positive, and of negative factors, negative (§ 11. Note 2); hence, an *odd root* (i. e. a root denoted by an odd number) of any quantity must have *the same sign as the quantity itself*. Thus,

$$(+a)^3=+a^3, \text{ and } (-a)^3=-a^3.$$

$$\therefore \qquad (+a^3)^{\frac{1}{3}}, \text{ or } \sqrt[3]{+a^3}=+a;$$

$$\text{and} \qquad (-a^3)^{\frac{1}{3}}, \text{ or } \sqrt[3]{-a^3}=-a.$$

f.) Again, since the product of an *even* number either of positive or of negative factors is always positive (§ 11. Note 2); therefore,

(1.) Every *even root* (i. e. every root denoted by an

(*d*) Lat. radix, *root*. (*e*) A modified form of the letter *r*, the initial of radix.

even number) of a *positive* quantity may be *either positive or negative.*

This character of the root is denoted by the *double sign* $\pm$ (read *plus or minus).* Thus,

$$(+a)(+a)=+a^2, \text{ and } (-a)(-a)=+a^2.$$

$$\therefore \qquad (a^2)^{\frac{1}{2}} \text{ or } \sqrt{a^2}=\pm a.$$

(2.) An *even root* of a *negative* quantity, can be *neither* positive *nor* negative, and therefore does not *really* exist, and is said to be IMAGINARY. For neither $(+a)(+a)$, nor $(-a)(-a)$ can produce $-a^2$.

§ 24. It is evident, from the definition of a *power*, that whatever has been demonstrated of quantities with exponents is true of powers. Hence we have the following rules.

RULE I.

a.) To *multiply* powers of the same quantity together.

Add their exponents. § 15. Cor. II.

$a^4.a^3=$ what? $a^{-5}.a^6$? $x^{\frac{3}{2}}.x^2$? $x^{\frac{3}{2}}.x^{-\frac{1}{2}}$? $3^4.3^{-3}$?

RULE II.

b.) To *divide* a power of a quantity, by any power of the same quantity.

Subtract the exponent of the divisor from that of the dividend. § 16. Cor. III.

$\dfrac{a^6}{a^3}=$ what? $\dfrac{a^6}{a^{-3}}$? $\dfrac{10^3}{10^0}$? $\dfrac{2^5}{2^4}$? $\dfrac{x^{\frac{3}{2}}}{x^{-\frac{1}{2}}}$? $\dfrac{x^{\frac{3}{2}}}{x}$?

RULE III.

c.) To find the *reciprocal* of a power.

Change the sign of the exponent. § 18.

What is the reciprocal of a? of a^4? of 10? of 10^2? of 10^{-1}? of R^2? of a^2x^{-2}? of $x^{\frac{3}{2}}$? of $a^{\frac{2}{3}}x^{-\frac{1}{3}}$?

RULE IV.

d.) To find any *power* of a power.

Multiply the exponent of the given power by that of the required power. § 20.

1. What is the second power of a^2? of $a^{\frac{1}{2}}$? of $16^{\frac{3}{2}}$? of $16^{-\frac{3}{2}}$? of $-a$?

2. $(a^2)^{-2} =$ what? $(a^{\frac{1}{2}})^6$? $(-10)^3$? $(a^{-4})^{\frac{1}{4}}$?

3. $(a^4)^{\frac{1}{2}} =$ what? $(10^6)^{\frac{3}{2}}$? $(R^2)^{-\frac{3}{2}}$? $(x^5)^{\frac{1}{5}}$? $(10^3)^{\frac{1}{3}}$?

§ 25. *e.*) The last rule obviously applies equally to the finding of a *root;* i. e. a power, whose exponent is *unity divided by the number* of the root (§ 23. *b*). But to multiply by such a fraction is the same as to divide by its denominator. Hence we have the common rule for finding a *root* of a power:

Divide the exponent of the power by the number of the root.

What is the third root of a^3? of a^2? of a? of 10^6?
What is the second root of 10^4? of x^3? of x^6? of 2?
What is the third root of $10^{\frac{3}{2}}$? of $a^{\frac{1}{2}}$? of $x^{\frac{3}{4}}$? of $x^{-\frac{1}{4}}$?

NOTE. It should be borne in mind, that the word *power* is used, in all these cases, in the *widest* sense; and that the rules are equally applicable to all the classes of powers specified in § 22.

§ 26. A quantity, *whose value is determined by the value assigned to another* quantity, is said to be a FUNCTION[f] of that other quantity.

Thus, a^2, a^3, a^4, are functions of a, because their value depends upon, and is determined by, the value assigned to

(*f*) Lat. functio, from fungor, *to perform*, as depending on the performance of certain operations upon another quantity.

a. Thus, let $a=1$, then $a^2=1$; if $a=2$, then $a^2=4$; if $a=10$, then $a^2=100$.

So, if $u=x^2$, or $u=2x$, or $u=3x$, then u is a function of x; or, as it is usually expressed, $u=F(x)$, or $u=f(x)$; where F and f are not factors, but mere abbreviations for the words *function of.*

A *power* is a *function of a quantity, expressed by an exponent* written over the quantity; i. e. an *exponential function* of the quantity.

§ 27. A power is said to be of such a DEGREE[g] as is indicated by the exponent. Thus,

a^3 is of the third degree; a^2, of the second; a of the first; a^{-4}, of the *minus* fourth; and $a^{\frac{1}{2}}$, of the one-half degree.

§ 28. The *degree of a term* is equal to the *sum of the exponents* of its literal factors. Thus,

a, x, $2x$, $3a^2x^{-1}$, $a^3b^0x^{-2}$ are of the first degree.

So $a^{\frac{1}{2}}x^{\frac{1}{2}}$, $a^{\frac{3}{2}}.x^{-\frac{1}{2}}$, $a^{\frac{2}{3}}x^{\frac{1}{3}}$ are of the first degree.

$2ax$, $2px$, y^2, a^4b^{-2}, $p^{\frac{1}{2}}x^{\frac{3}{2}}$ are of the second degree.

$3a^2x$ is of the *third*, and $4a^3x$, of the fourth degree.

$a^{\frac{1}{2}}$ is of the one half, and $a^{\frac{2}{3}}$, of the two thirds degree.

$a^{-2}x^{-2}$, and a^3x^{-7} are of the *minus* fourth degree.

1. $9a^5b^4c^{-3}$ is of what degree? $15x^2y^2$? $5a^3b\times 6xy$? $a^{\frac{5}{2}}b^{\frac{3}{2}}x$? $3a^{\frac{2}{3}}x^{\frac{7}{3}}$? $a^{\frac{5}{3}}x^{-\frac{2}{3}}$? $a^{\frac{1}{3}}x^{-\frac{2}{3}}$? $a^{-3}x^0$?

NOTE. A term is also sometimes said to have as many *dimensions*[h] as there are units in its degree.

(*g*) Fr. degre', from Lat. gradus, *step*. (*h*) Lat. dimensio, from dimetior, *to measure*. The use of this word resulted from taking a factor to represent a line, and, consequently, a product of two factors to represent a surface, and one of three factors, to represent a solid. The factors were therefore regarded as the *dimensions*, or measures of the magnitudes. See Geom. §§ 3, 170,177. The word is, of course, not strictly applicable to any term of a degree higher than the third (Geom. § 2. *a*), or lower than the first.

a.) In estimating the degree of a *fractional* term, the exponents of the letters in the denominator must, of course, be regarded as negative (§ 14, 16), and subtracted from the sum of the exponents of the letters in the numerator. Thus, $\frac{x^3}{a^2}$ is of the first degree; $\frac{a^2b^2}{a^2}$ and $\frac{abx}{c}$, are of the second.

b.) A term is said to be of the first, second, third or *n*th degree *with respect to a particular letter or letters*, when it contains the first, second, third, or *n*th degree of the letter or letters. Thus,

$3a^2x$, and $a^{-1}x$ are of the first degree with respect to x.
b^2x^2 and ax^2 are of the second " " x.
$abx^{\frac{1}{2}}$ and $\sqrt{x}$ are of the one half " " x.
a^2x^0, and abc are of the zero " " x.

axy is of the first degree with respect to either a, x or y; and of the second degree with respect to x *and* y, or any two of the letters; while it is of the third degree with respect to all the letters.

§ 29. Terms of *the same degree* are said to be HOMOGENEOUS[i].

Thus, y, $2x$, and $\frac{1}{2}a^2x^{-1}$ are homogeneous. So a^3, $3ax^2$, xyz; in like manner, y, $\frac{ab}{c}$, and $\frac{B^2x''}{A^2y''}x$.

1. Are A^2y^2 and B^2x^2 homogeneous? x^3, $2y^2$ and x? R^2 and $\sin a \sin b$?

§ 30. Terms, which consist of *the same literal factors, with the same exponents* (i. e. each letter being of *the same degree in the several terms*), are called SIMILAR or LIKE terms.

Thus, $2xy$, $8xy$, and $3yx$ are similar terms; so $3x^2y$, and $\frac{1}{2}x^2y$. But $3x^2y$ and $3xy^2$ are *not* similar, because, though the letters are the same, they have different exponents in the two terms. Are $3x^2y$ and $3xy^2$ homogeneous?

(*i*) Gr. ὁμογενής, compounded of ὁμός, *like*, and γένος, *kind*.

Are a^2b^2 and x^2y^2 similar? Are they homogeneous?

a.) Thus terms may be homogeneous without being similar, but they cannot be similar without being homogeneous.

b.) Terms, in which the same letter, with the same exponent, enters, are sometimes said to be *similar with respect to that letter.* Thus the terms ax, $abcx$ and c^4x are similar with respect to x.

MONOMIALS AND POLYNOMIALS.

§ 31. A quantity consisting of *one term*, is called a MONOMIAL[k]; of *more than one*, a POLYNOMIAL[l]. A polynomial of *two terms* is called a BINOMIAL[m]; one of *three* terms, a TRINOMIAL[n].

Thus, $2ax$, a, a^2b^2, abc are monomials; so $ab\times xy\div z$; $a+b$, $a-b$, x^2-y^2 are binomials; $a+b+c$, $a^2\pm 2ax+x^2$ are trinomials.

§ 32. A *polynomial* is said to be *homogeneous*, when *all its terms are homogeneous* (§ 29.).

Thus, $a^3\pm 3a^2b+3ab^2\pm b^3$, $A^2y^2+B^2x^2-A^2B^2$ are homogeneous polynomials.

1. Is $x^2+y^2-R^2$ homogeneous? $x^5\pm 5x^4y+10x^3y^2\pm 10x^2y^3+5xy^4\pm y^5$?

2. Is a^2+b^3 homogeneous? $a^2+\dfrac{b^3}{a}$? $a^3+\dfrac{b^3}{a}$?

§ 33. When the several terms of a polynomial contain different powers of any letter or letters, it is generally convenient to *arrange* the terms according to the powers of some one letter; that is, to write the term containing either the highest, or the lowest power of the letter first, and the other terms successively, according to the order of their ex-

(*k*) monome, from Gr. *μόνος*, *alone*, and *ὄνομα*, Lat. nomen, *name*; as being expressed by a single name or term. (*l*) polynome from Gr. *πολύς*, *many*, and *ὄνομα*, *name*. (*m*) Lat. bis, *twice*, and nomen, *name*. (*n*) Lat. tres, *three*, and nomen, *name*.

ponents; from highest to lowest, or from lowest to highest. If the highest exponent is placed first, the terms are said to be arranged in a *descending series*, or according to the descending powers of the letter; if the lowest is placed first, the arrangement is said to be in an *ascending* series, or according to the ascending powers of the letter.

Thus, $a^2+2ab+b^2$ is arranged according to the ascending powers of b, and according to the descending powers of a.

1. Arrange $3a^2b+3ab^2+b^3+a^3$ according to the descending powers of a; of b.

2. Arrange $qx^{n-2}+x^n+px^{n-1}+rx^{n-3}$ according to the descending powers of x.

NOTE. The letter, according to whose powers the terms of a polynomial are arranged, is frequently called the *letter of arrangement*. When there is no special reason for a different order, it is generally convenient to write the letters of each term in the order of the alphabet; and also to take the first of those letters, as the letter of arrangement.

REDUCTION OF POLYNOMIALS.

§ 34. A polynomial, which *contains similar terms*, can be *reduced to a simpler form.*

This is done according to the principles of § 4. Thus, in the polynomial $4a-6a+9a-3a$, $4a$ and $9a$ are to be added, and $6a$ and $3a$ are to be subtracted. It is usually most convenient to bring together the terms which are to be added, and also the terms which are to be subtracted, and then take the less from the greater. If the quantity to be added is greater than that to be subtracted, the result is to be added; i. e. is positive. If the quantity to be subtracted is greater than that to be added, the result is to be subtracted; i. e. is negative (§ 4. a, b). Hence, for reducing or simplifying a polynomial containing similar terms, we have the following

RULE.

Add together the coefficients of such similar terms as have the sign +; and then the coefficients of such as have the sign —; take the less of these sums from the greater, and prefix the remainder, with the sign of the greater, to the common letter or letters. Thus,

$$4a-6a+9a-3a=4a+9a-6a-3a=13a-9a=4a.$$

a.) Terms of a polynomial, which are not similar, will, of course, remain as they were; each being preceded by its own sign.

Reduce the following polynomials to their simplest form.

1. $a^2-ab-ab+b^2$. *Ans.* $a^2-2ab+b^2$.
2. $a^2+ab-ab-b^2$.

b.) There may be several sets of similar terms in the same polynomial. In that case, the above method must, of course, be applied to each set separately. Reduce,

1. $5a+6b-7x-8b+3a-4a+2x+9a-3x$.
2. $a^4-3a^3x+3a^2x^2-ax^3-a^3x+3a^2x^2-3ax^3+x^4$.
3. $1+x-1+x$. 4. $1+x+1-x$.
5. $y^2+x^2-px+\frac{1}{4}p^2-x^2-px-\frac{1}{4}p^2$.
6. $2bx+2x^2-b^2-2bx-x^2$.
7. $a^3+a^2b+ab^2-a^2b-ab^2-b^3$.

c.) If a polynomial contains several terms *similar in respect to a certain letter* (§ 30. *b*), the same principle will obviously apply. Thus, the terms $ax+bx-2cx$, are similar in respect to x. Now, a times x, *plus* b times x, *minus* $2c$ times x is evidently the same as x taken $a+b-2c$ times, which (§ 2. *h*) is expressed $(a+b-2c)x$. Hence, we may write the coefficients, whether numerical or literal (§ 9. *b*), of the common letter or letters in the several terms, in order, with the signs of the terms; enclose the whole expression, so formed, in a parenthesis, or put it under a vinculum;

and write the common letter or letters, without the parenthesis or vinculum, as a separate factor. Reduce,

1. $A^2x^2-c^2x^2$. *Ans.* $(A^2-c^2)x^2$.
2. $2px''+px-px''$.
3. $A^2y^2+B^2x^2-A^2B^2-A^2y''^2-B^2x''^2+A^2B^2$.

§ 35. The *numerical value* of an algebraic expression is the result obtained by assigning particular values to the letters, and performing the operations indicated by the symbols. Thus,

Let $a=10$, and $b=5$, then $a+b=15$; $(a+b)^2=15^2=225$; $a^2+2ab+b^2=10^2+2.10.5+5^2=225$.

1. Let $a=10$ and $b=4$, and find the value of $a^3+3a^2b+3ab^2+b^3$; $(a-b)^2$; a^2-b^2; $(a+b)(a-b)$.

2. Find the value of the same expressions, when $a=8$, and $b=3$; when $a=20$, and $b=5$; when $a=10$, and $b=10$; when $a=10$, and $b=9$; when $a=1$, and $b=1$.

3. Find the value of $y-2x-4$, when $y=10$, and $x=3$; when $y=8$, and $x=2$; when $y=4$, and $x=0$.

EQUATIONS.

§ 36. The expression of equality between two quantities constitutes an EQUATION[o]; as,

$$5+4=10-1;\ a^m\times a^n=a^{m+n};\ 3x=15;\ ax=b.$$

a.) The two quantities themselves are called the MEMBERS[p] or SIDES of the equation. The member on the left of the sign is styled the *first*, and that on the right, the *second* member.

b.) Most of the investigations and reasonings of Algebra are carried on by means of equations.

§ 37. *c.*) The simplest form of equation is that, in which

(*o*) Lat. æquatio, from æquo, *to make equal*. (*p*) L. membrum, *limb*.

the two sides are precisely alike; as $10=10$; $x+2=x+2$. These are called *identical*[q] equations.

d.) Another class of equations we have already employed, in expressing the results of operations, or the truth contained in such results. Thus, $a^3 \times a^4 = a^7$; $a^m a^n = a^{m+n}$; $1 \div x = x^{-1}$ (§ 17). These may be called *absolute* equations; inasmuch as their truth has no dependence upon the value assigned to a, x, m or n. The second member necessarily results from the operation indicated in the first.

§ 38. *e.*) In another class of equations, there is no *absolute* and essential equality between the members; but they are equal only on the *condition*, that some particular value or values be given to one or more of the quantities involved. Equations of this kind may be called *conditional* equations. Thus, $2x=10$ is a *conditional* equation, in which the equality of the members depends on the *condition*, that x shall be equal to 5. If 4 were taken as the value of x the two members would not be equal; we should have 8 on one side, and 10 on the other. But taking $x=5$, then $2 \times 5 = 10$, or $10=10$.

f.) A conditional equation, moreover, itself furnishes the means of investigating and ascertaining the value which must be given to x, in order that the members may be equal; that the equation may become absolute or identical. For $1x$ is obviously half as much as $2x$; if then, we have

$$2x=10,$$

we shall have $\qquad x = \frac{1}{2}$ of $10 = 5$, the necessary value of x, as above.

Conditional equations may therefore be called equations of *investigation*.

§ 39. Any quantity, to which a particular value must be given, in order to render the members equal, is called an *unknown* quantity (§ 1. *c*, N). That *value* of an unknown quantity, *which renders the members equal*, is called a ROOT

(*q*) Fr. identique, from Lat. idem, *the same*.

of the equation. When this value is substituted for the unknown quantity, it is said to *satisfy* or *verify* the equation. The process of finding the root of an equation is called *solving* the equation.

NOTE. When *equations*, without any specification, are spoken of, or when the subject of equations is spoken of, as a branch of algebraic science, the expression must, in general, be understood to imply *conditional* equations.

§ 40. Conditional equations are distinguished into *orders*, according to their *degree*.

a.) The *degree of an equation* depends on the degrees of its terms with respect to the unknown quantity or quantities (§ 28. *b*); and is determined by the *range of those degrees from lowest to highest.*

b.) The full consideration of this subject would involve the consideration of equations containing negative and fractional powers of the unknown quantities.

c.) For the present, however, it is sufficient to consider those equations only, in which the exponents of the unknown quantity or quantities are *all integral*, and in which *the least* of those exponents is *zero*.

d.) In this case, the degree of the equation is the same as the *highest degree of its unknown quantity or quantities*. Thus,

$ax=b$, $2x=10$, and $x+y=10$ are of the first degree.

$ax^2=b$, $x^2+3x=10$, and $xy=20$ are of the second degree.

§ 41. We shall, at present, confine ourselves to the consideration of equations containing but *one* unknown quantity; subject also to the limitation mentioned above (§ 40. *c*).

These equations are said to be of the same degree as the *highest power of the unknown quantity* which they contain. Thus,

$3x=18$; $ax=b$ are equations of the first degree.

$x^2=9$; $ax^2+bx=c$ are equations of the second degree.
$ax^3+bx^2+cx=h$; $x^3=8$ " third "
$ax^4+bx^3=c$; $x^4=16$ " fourth "
$x^n+px^{n-1}+$&c.$=h$ is " nth "

NOTE. Equations of the *first* degree are sometimes called *simple* equations; those of the *second* degree, *quadratic*[s]; those of the *third*, *cubic*; and those of the fourth, *biquadratic*[t].

§ 42. All reasoning by means of equations proceeds upon a single AXIOM[u], or SELF-EVIDENT TRUTH; viz. EQUAL QUANTITIES, EQUALLY AFFECTED, REMAIN EQUAL. Geom. 20.

The meaning of this axiom, which, though not always expressed in words, is assumed in all mathematical operation, may be illustrated by a few familiar examples. Thus, $3\times5=15$ is an equation. Adding 2 to both sides, we have $3\times5+2=15+2$. Subtracting 4 from both sides of the first equation, we have $3\times5-4=15-4$. In like manner, we might multiply or divide both sides by the same quantity, and obtain equal products or quotients.

Hence, *if both members of an equation be*

a. increased by the addition of,	*equal quantities, the results will be equal.*
b. diminished by the subtraction of,	
c. multiplied by,	
d. divided by,	

§ 43. I.) 1. Given $x-3=7$; to find the value of x.

Add 3 to each side;

then $x-3+3=7+3$. § 42. *a.*

or $x=10$, the value required.

2. Given $x-5=4$, to find the value of x.

Ans. $x=9$.

3. Given $x-16=20$, to find the value of x.

(*s*) Lat. quadra, *square*. (*t*) Lat. bis, *twice*, and quadra, *square*.
(*u*) Gr. ἀξίωμα, from ἀξιόω, *to deem worthy, suppose, take for granted.*

II.) 4. Given $x+3=7$, to find x.

Subtract 3 from each side;

then $$x+3-3=7-3. \qquad \S\ 42.\ b.$$

or $$x=4, \qquad \text{the value required.}$$

5. Given $x+10=15$, to find x. *Ans.* $x=5$.

6. Given $2x=10+x$, to find x.

Subtract x from each side;

then $$2x-x=10+x-x. \qquad \S\ 42.\ b.$$

or $$x=10, \text{ the value required.}$$

7. Given $3x-10=10+2x$, to find the value of x.

NOTE. To verify or *prove* these results, we have only to introduce, into the given equation, the value found for the unknown quantity in place of the unknown quantity itself. Thus, in example 1 above, substituting for x its value found, we have

$$10-3=7, \text{ an } \textit{absolute equation}. \quad \text{See § 39.}$$

Verify the otber equations in like manner.

§ 44. Thus we see that the application of § 42. *a* and *b* causes any *term*, which stands on one side of an equation, preceded by the sign either of addition or subtraction, to disappear from that side, and to reappear on the other side with the opposite sign. Thus, in § 43. 1, by adding 3 to both sides, and reducing, -3 is canceled in the first member, and $+3$ appears in the second; so, in § 43. 4, $+3$ is canceled in the first member, and -3 appears in the second.

This is called *transposition*[v]. For the same effect would obviously have been produced, if we had simply removed the term from the one side, and written it with the opposite sign upon the other. In fact, removing -3 (i. e. ceasing to subtract 3) from the first member (§ 43. 1) *increases* that member by 3; 3 must, therefore, be *added* to the second member, to preserve the equality. So (§ 43. 4), removing $+3$ (i. e. ceasing to add 3) *diminishes* the first

(*v*) Lat. transpositio, from transpono, *to place beyond, carry over.*

member by 3; 3 must, therefore, be *subtracted* from the second member. Hence,

Any quantity may be transposed from one side of an equation to the other, if, at the same time, we change its sign.

a.) If we transpose *all the terms* of an equation, the signs will all be changed, and the members will still be equal. Hence,

Corollary. *The signs of all the terms of an equation may be changed at pleasure, without affecting the equality of the members.*

b.) It is also evident from § 42. *a*, *b*, that *the same quantity, with the same sign, occurring on both sides of an equation, may be suppressed.*

c.) The object of transposition is, in general, to bring all the terms containing the unknown quantity to stand on one side of the equation; and all the known terms, upon the other. The polynomials so formed should, of course, be reduced to their simplest form (§ 34).

1. Given $8x+4=7x+12$, to find the value of x.

2. Given $15y-3=12+5y-3+9y$, to find the value of y. *Ans.* $y=12$.

3. Given $2x+a+b=3x+2a-2x$, to find x. *Ans.* $x=a-b$.

4. Given $4x+3a+2b=4a+3x+b$, to find x.

5. Given $3x-10=5+2x-15$, to find x. *Ans.* $x=0$.

6. Given $2x-10=x-15$, to find x. *Ans.* $x=-5$.

§ 45. 1. Given $\frac{x}{4}+3=8$, to find x.

Transpose 3;

then $$\frac{x}{4}=5. \qquad \text{§ 44.}$$

Multiply by 4;

then $$x=20. \qquad \text{§ 42. } c.$$

To verify this equation, substitute 20 for x, and we have

$$\frac{20}{4}+3=8;\ \text{or } 5+3=8, \text{ an absolute equation.}$$

2. Given $\frac{x}{3}-5=3$, to find x. *Ans.* $x=24$.

3. Given $\frac{x}{2}-\frac{1}{3}x-2=0$, to find x.

Multiply by 2;

then $x-\frac{2}{3}x-4=0$. § 42. *c*.

Multiply by 3;

then $3x-2x-12=0$. § 42. *c*.

Reducing, $x-12=0$. § 34.

$\therefore$ $x=12$. § 44.

4. Given $\frac{x}{3}-\frac{x}{4}=5$, to find x.

§ 46. Thus, if a quantity in an equation be divided by any number, the application of § 42. *c* enables us to free it from its divisor, i. e. *to clear the equation of fractions.*

The terms of an equation may, therefore, be freed from divisors, or, in other words, an equation may be *cleared of fractions*, by *multiplying all the terms of the equation by the denominators of the fractional terms.*

NOTE. The equation is to be multiplied first by one of the denominators, and then the resulting equation by another, and so on, till all the terms containing the unknown quantity become whole numbers. In this process improper fractions may always be reduced to whole numbers, whenever it can be done; and no more multiplications should be performed, than are necessary to clear the equation of fractions.

a.) The same effect would obviously be produced by multiplying all the terms of the equation, by any common multiple of the denominators; i. e. by any number which the denominators will all divide without a remainder. For if a denominator will divide the multiplier, it will necessarily

divide the product of its own numerator into that multiplier. Thus,

Let $$\frac{x}{2}+\frac{x}{5}+\frac{x}{6}=\frac{5x}{6}+1.$$

Multiply by 30;

then $$15x+6x+5x=25x+30. \qquad \text{§ 42. } c.$$

$$x=30. \qquad \text{§§ 44, 34.}$$

Given $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-\frac{x}{6}=\frac{7x}{8}+1$, to find x.

NOTE. A common multiple may easily be found by trial. Thus, in the above equation, try 8, the largest denominator, and see if the other denominators will divide it without a remainder. We find, that 3 and 6 will not so divide it. Then multiply it by 2; still we do not obtain a multiple of those denominators. Multiply 8 by 3, and all the denominators will divide the product; 24, therefore, is a multiplier, which will clear the equation of fractions. It is important to employ the smallest multiplier, which will accomplish the object.

b.) By clearing of fractions, the coefficients of the unknown quantity all become integral; and the polynomial, formed by collecting all the terms containing the unknown quantity into one member, is the more easily reduced to a simpler form (§ 34).

NOTE. Whether transposition or clearing of fractions be first performed, is indifferent. Any course may be taken in this respect which is found convenient. See § 45. 1, 3. The whole process of clearing of fractions, transposition, and reducing the polynomial members to their simplest form, is sometimes called the *reduction* of the equation.

1. Given $\frac{2x}{3}+\frac{3x}{4}=\frac{4x}{3}+5$, to find x. *Ans.* $x=60$.

2. Given $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+4$, to find x.

3. Given $x+10=\frac{5x}{9}+\frac{x}{3}+50$, to find x.

§ 47. 1. Given $2x-7=9-6x$, to find x.

Transpose;

then $$8x=16. \qquad \text{§ 44.}$$

Divide the terms by 8;

then $x=2$. § 42. *d.*

2. Given $3x+5=x+20$, to find x. *Ans.* $x=7\frac{1}{2}$.

3. Given $4x-8=40-2x$, to find x.

Thus it is obvious, that, when, by any means, a single term containing the unknown quantity is made to constitute one member of an equation, while the other member consists wholly of known quantities, the *root* of the equation will be found by *dividing both members by the coefficient of the unknown quantity.*

NOTE. If the coefficient is unity, there will, of course, be no need of dividing.

§ 48. Bringing together the principles above explained (§§ 43–47), we have, for solving equations of the first degree, containing but one unknown quantity, the following

RULE.

Clear the equation of fractions, and bring all the terms containing the unknown quantity upon one side, and all the known terms upon the other. Reduce the two members to their simplest form, and divide them both by the coefficient of the unknown quantity.

1. Given $\frac{x}{2}+2=\frac{x}{4}+\frac{x}{5}+3$, to find x. *Ans.* $x=20$.

2. Given $6\frac{1}{8}+\frac{3x}{2}-3=\frac{7x}{6}+\frac{3x}{8}+2\frac{11}{16}$, to find x.

$6\frac{1}{8}$ and $2\frac{11}{16}$ are obviously the same as $6+\frac{1}{8}$ and $2+\frac{11}{16}$. Either form may be used. In this instance, the latter form will be found more convenient for reduction.

3. Given $x-\frac{3}{7}x=33-3x$, to find x.

§ 49. Many equations, which are not of the *first degree*, can be so easily reduced to that form, that they may, properly enough, be briefly considered in this place.

I. An equation may contain higher or lower powers of the unknown quantity, which may be *canceled* by transposition, so as to leave no power higher than the first, or lower than the zero power. Thus,

Let $$x^n+\frac{x}{3}+\frac{3x}{5}+3=x+x^n.$$

Canceling x^n, we have

$$\frac{x}{3}+\frac{3x}{5}+3=x, \text{ an equation of the first degree.}$$

Equations of this form, or which, on reduction, take this form, need no farther remark.

§ 50. II. An equation may contain only the zero and *minus* first powers of the unknown quantity. This may properly be called an equation of the *minus* first degree. But, if we multiply by the unknown quantity, we shall evidently reduce the equation to the common form of the first degree. Thus,

Let $$x^{-1}+2x^{-1}+3x^{-1}=2,$$

or $$6x^{-1}=2.$$

Multiply by x;

then $$6x^0=2x;\ \text{or } 6=2x. \qquad \S\ 42.\ e.$$

$$\therefore \qquad x=3.$$

Otherwise, $$\frac{1}{x}+\frac{2}{x}+\frac{3}{x}=2. \qquad \S\ 17.$$

Clearing of fractions, $1+2+3=2x$; or $6=2x$. § 46.

$$\therefore \qquad x=3.$$

Hence,

Equations of the MINUS *first degree* can be reduced to the first degree *by multiplying by the unknown quantity.*

§ 51. III. Any equation containing *only two* powers of the unknown quantity, provided their *exponents differ by*

unity, may evidently be reduced to the common form of the first degree, by *dividing by the lowest power* of the unknown quantity. Thus,

Let $x^2-10x=0$; or $x^2=10x$.

Divide by x;

then $x-10=0$; or $x=10$. § 42. *d.*

So also if $x^n=5x^{n-1}$,

then, dividing by x^{n-1}, $x=5$.

1. Given $3x^5+2x^4-x^5=\frac{1}{2}x^5+11x^4$, to find x.

2. Given $x^{\frac{3}{2}}+2x^{\frac{1}{2}}=\frac{1}{3}x^{\frac{3}{2}}+5x^{\frac{1}{2}}$, to find x.

a.) The principle of § 51 obviously includes that of § 50, inasmuch as dividing by x^{-1} is the same as multiplying by x.

b.) The whole class of equations included under § 51, are actually of the first degree, according to the more general definition of the degree of an equation. For, the *range of the degrees* of the terms with respect to the unknown quantity, *from lowest to highest*, is expressed by unity (§ 40. *a*); as is found by subtracting the lowest from the highest.

§ 52. IV. An equation may contain, besides the zero power of the unknown quantity, only a simple *root* of the unknown quantity; i. e. it may contain only the zero and the one half, one third, or $\frac{1}{n}$th powers of the unknown quantity. The equation, in this case, is of the one half, one third, or $\frac{1}{n}$th degree. Thus,

Let $x^{\frac{1}{2}}=5$; or $\sqrt{x}=5$.

Squaring both members (§ 42. *c*),

we have $x=25$.

So, if we had $x^{\frac{1}{n}}=a$, or $\sqrt[n]{x}=a$, we should find $x=a^n$. Hence,

An equation of the $\frac{1}{n}$th degree can be reduced to the first degree, *by raising both members to the nth power.*

NOTE. This operation evidently comes under § 42. *c*, for the members being equal, multiplying them by themselves is multiplying them by equal quantities. So, if they be separated into the same number of equal factors, one factor on one side will be equal to one on the other; i. e. any *fractional* power or *root* of one side is equal to the same power or root of the other. For this is formed by dividing all the factors but one out of each member (§ 42. *d*). Hence,

If both members of an equation be raised to the same power, whether integral or fractional (§ 22), *the results will be equal.*

1. Given $\frac{1}{2}x^{\frac{1}{3}}+2=\frac{1}{3}x^{\frac{1}{3}}+3$, to find x.
2. Given $y^{\frac{1}{4}}+2=\frac{\sqrt[4]{y}}{3}+3$, to find y.

§ 53. We have classed equations with reference to their *unknown* quantities. They are also sometimes distinguished, with reference to the form in which their *known* quantities are expressed, as *numerical* or *literal.*

A *numerical* equation is one, in which *the known quantities are all expressed by numbers;* as $x^2=10x+24$.

A *literal* equation is one, in which *a part* or *all* of the *known quantities are expressed by letters;* as $ax^2+2bx=c$.

§ 54. A conditional equation is the algebraic expression of a PROBLEM[w]; i. e. something *proposed to be performed* or *discovered.*

Thus, the equation $x-3=7$ (§ 43. 1), proposes this problem; viz. To find a number such that if it be diminished by 3, the remainder shall be 7.

So the equation $\frac{1}{4}x+3=8$ (§ 45. 1), proposes this problem; viz. To find a number, whose fourth part, increased by 3, is equal to 8.

State, in like manner, the problems involved in each of the equations of §§ 43–52. Compare § 3. *a.*

§ 55. As we have seen, an equation is the algebraic expression of a problem; and the solution of the equation gives the solution of the problem. Hence to solve a *prob-*

(*w*) Gr. πρόβλημα, from προβάλλω, *to throw* or *lay before.*

lem, we have only to *express its conditions algebraically by an equation*, and then *solve the equation.*

The process of expressing the conditions of a problem by an equation is sometimes called *putting the problem into an equation;* and is frequently more difficult than the subsequent solution of the equation.

The student will most readily learn the methods of *forming an equation* from a problem, by stating the problems involved in the preceding equations, and observing how the conditions of each problem are expressed in the equation. He will find, that the process conforms, in general, to the following

RULE.

Represent the unknown quantity by some letter, as x; then combine the known and unknown quantities according to the conditions of the problem. The result will be an equation expressing those conditions. See § 3. *b.*

In this process, we treat the unknown quantity as if it were known; and perform upon it just those operations which would be necessary to prove the correctnesss of the result, if we had fixed upon a value for the unknown quantity. We have, in fact, fixed upon a representative of that value, in the letter which we have chosen to denote the unknown quantity.

Problem 1. To find a number whose fifth part exceeds its sixth part by 10.

Let x represent the number sought.

Then $\frac{1}{5}x$, or $\frac{x}{5}$ will represent its fifth part.

and $\frac{1}{6}x$, or $\frac{x}{6}$ will represent its sixth part.

Then, by the condition,

$$\frac{x}{5}-\frac{x}{6}=10.$$

$$\therefore \qquad 6x - 5x = 300. \qquad \text{§ 46.}$$

or
$$x = 300.$$

Verification, $\frac{300}{5} - \frac{300}{6} = 60 - 50 = 10.$

Prob. 2. What sum of money is that, whose fourth part exceeds its fifth part by 5 dollars?

Prob. 3. What sum of money is that, whose fourth part exceeds its fifth part by a dollars?

Let $x =$ the sum.

Then
$$\frac{x}{4} - \frac{x}{5} = a.$$

$\therefore$ (§46) $5x - 4x = 20a$; or $x = 20a$, the sum.

NOTE. The last problem, it will be seen, is the same as the preceding, except that the difference between the fourth and fifth parts of the number is denoted by a, which may represent *any number whatever.* This is called a *general* solution, or *generalization* of the problem. In this solution, a, the given excess of the fourth part above the fifth, *remains in the result;* whence we learn, that the whole number must be 20 times that excess. Thus, if that excess be 1, the number must be 20; if the excess be 2, the number must be 40; if the excess be 5, the number will be 100; &c.

Prob. 4. A, B, and C enter into partnership. A contributes a certain sum; B contributes three times, and C, four times as much as A. Their whole stock is \$20,000. How much did each contribute?

Let $x =$ A's part; then $3x =$ B's, and $4x =$ C's.

$$\therefore \qquad x + 3x + 4x = 20{,}000.$$

Prob. 8. A man and boy work together, for \$75. The man's work is worth four times as much as the boy's. How shall they divide the money?

CHAPTER I.

ADDITION AND SUBTRACTION.

X

I. ADDITION.

§ 56. ADDITION is the process of finding the *aggregate* of several quantities. See § 8.

Adding quantities is bringing them together, so that each may have its proper effect in making up the aggregate; those which increase, and those which diminish the amount, being characterized, each by the proper sign. See § 4. Hence, for adding quantities, we have the following

RULE.

§ 57. *Write the quantities to be added, one after another, each with its own sign.*

a.) If the polynomial, thus formed, contain similar terms, it may, of course, be reduced by § 34.

b.) This reduction can often be easily performed without first writing out all the terms at full length. For this purpose, there is an advantage in writing the similar terms under one another. Thus,

Add $a^4+2a^3y+a^2y^2$, $-2a^3y-4a^2y^2-2ay^3$, and $a^2y^2+2ay^3+y^4$.

Writing these expressions with their similar terms under one another, we have,

$$\begin{array}{llll} a^4+2a^3y+a^2y^2 & & & \\ \quad -2a^3y-4a^2y^2-2ay^3 & & & \\ \qquad\qquad a^2y^2+2ay^3+y^4 & & & \\ \hline a^4 \qquad -2a^2y \qquad +y^4 & & & \end{array}$$

1. Add $a+b$ to $c+x$. *Sum*, $a+b+c+x$.

2. Add $a+b$, and $a-b$. *Sum*, $2a$.

3. " $\frac{1}{2}a+\frac{1}{2}b$, and $\frac{1}{2}a-\frac{1}{2}b$. *Sum*, a. That is, *the half sum* + *the half difference* of any two quantities = *the greater.*

4. Add a^2+ab, and $ab+b^2$.

5. " a^2-ab, and $ab-b^2$.

6. " $a^3-2a^2b+ab^2$, and $a^2b-2ab^2+b^3$.

7. " $x^3+2x^2y+xy^2$, and $x^2y+2xy^2+y^3$.

8. " $y^2+x^2-px+\frac{1}{4}p^2$, and $-x^2-px-\frac{1}{4}p^2$.

9. " $A^2+2cx+\dfrac{c^2x^2}{A^2}$, and $A^2-2cx+\dfrac{c^2x^2}{A^2}$.

10. " $y^2+x^2+2cx+c^2$, and $y^2+x^2-2cx+c^2$.

11. " $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\dfrac{x^5}{5}-\dfrac{x^6}{6}+\dfrac{x^7}{7}-\dfrac{x^8}{8}+\dfrac{x^9}{9}$, and $-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\dfrac{x^4}{4}-\dfrac{x^5}{5}-\dfrac{x^6}{6}-\dfrac{x^7}{7}-\dfrac{x^8}{8}-\dfrac{x^9}{9}$.

12. Add $a^2+2ab+b^2$, and $a^2-2ab+b^2$.

13. " $\sin a \cos b+\cos a \sin b$, to $\sin a \cos b-\cos a \sin b$.

14. " $2a+a^2x^{-1}$, $3a^{\frac{3}{2}}x^{-\frac{1}{2}}$, $6ax^0$, $10a^{\frac{1}{2}}x^{\frac{1}{2}}$, $-15a^0x$, $-12a^{-\frac{1}{2}}x^{\frac{3}{2}}$, $9a^{-1}x^2$, $10a^2x^{-1}$, $11a^{\frac{3}{2}}x^{-\frac{1}{2}}$, $8ax^0$, $-5\sqrt{a}\sqrt{x}+5a^0x+2a^{-\frac{1}{2}}x^{\frac{3}{2}}$, and $18a^0x$.

15. " $ay+bx$, and $a'y-b'x$.

Ans. $(a+a')y+(b-b')x$, or $\left.\begin{array}{r} a \\ +a' \end{array}\right| y + \left.\begin{array}{r} b \\ -b' \end{array}\right| x.$

16. " $ay-bx+cz$, $a'y+b'x-c'z$, and $-a''y+b''x-c''z$.

17. " y^3+ay^2+aby; by^2+acy, and $cy^2+bcy+abc$; and arrange the result according to the descending powers of y (§§ 33, 34. c).

18. Add, member by member (Geom. § 22), the equations $-7x+5y=19$, and $10x-5y=-10$.

Ans. $3x=9$. $\therefore x=$ what?

PROBLEMS.

§ 58. 1. The sum of $2x-10$ and $4x-20$ is equal to $3x$. What is the value of x?

$$2x-10+4x-20=3x.$$

$$\therefore \quad 6x-30=3x. \quad \therefore 6x-3x=30;$$

or $$3x=30. \quad \therefore x=10.$$

2. The sum of $5x-8$, $2x-20$, and $x-10$ is equal to $10-4x$. What is the value of x?

3. The sum of $\frac{1}{2}x-1$, $2-\frac{3}{4}x$, $1+x-\frac{1}{3}x$, and $x-2$ is equal to $x+5$. What is the value of x?

4. The sum of $2x$, $7x$, $\frac{3}{2}x$, and -6 is -23. What is the value of x?

5. The sum of $13\frac{3}{4}-\frac{1}{2}x$ and $-2x+8\frac{3}{4}$ is nothing. What is the value of x?

$$13\tfrac{3}{4}-\tfrac{1}{2}x-2x+8\tfrac{3}{4}=0. \therefore 22\tfrac{1}{2}=2\tfrac{1}{2}x. \therefore x=9.$$

6. A's property is $3a$ dollars, and his debts $2a$; B's property is $5a$, and his debts $3a$; if they make common stock of their property, what is their net capital, x?

Let $a=100$, 500, 1000, 10,000, and find the value of x in each case.

7. An estate was divided among three sons. The eldest received \$4000 less than one half; the second received one third; and the youngest received \$2000 more than one quarter of the whole. What was the estate, and what did each receive?

Let the estate be represented by x. Then we shall have,

the share of the first $=\dfrac{x}{2}-4000$;

" second $=\dfrac{x}{3}$, and

share of the third $=\frac{x}{4}+2000$. The sum of the shares is, of course, equal to x.

8. Let the first receive a less than half; the second, one third; and the youngest, one half of a more than one quarter of the estate. What was the estate, and what did each receive?

Here the shares are $\frac{x}{2}-a$, $\frac{x}{3}$, and $\frac{x}{4}+\frac{a}{2}$.

$$\therefore \quad \frac{x}{2}-a+\frac{x}{3}+\frac{x}{4}+\frac{a}{2}=x.$$

$$\therefore \quad x=6a, \text{ the estate.}$$

Let $a=1000$, 100, 2000, 10,000, and find the value of the estate, and the share of each.

9. A, B and C, form a partnership; A puts in a certain amount of stock; B puts in \$2000 less than the double of A's; and C invests \$8000 less than the triple of A's. The whole stock is \$50,000. Required each one's share.

10. Suppose the sum of the distances of Mercury, Venus, and the Earth from the Sun is, in round numbers, 200 millions of miles; and that the distance of Mercury is 31 millions less, and of the Earth 58 millions more than that of Venus. What are their several distances?

Let $x=$ the distance of Venus.

II. SUBTRACTION.

§ 59. SUBTRACTION is the process of finding the *difference* between two quantities. See § 8.

a.) We have seen (§ 7. *b*), that the subtraction of a negative quantity has the same effect as the addition of an equal positive quantity. Therefore, to subtract a negative quantity, we have only to change its sign and add it.

b.) It is also perfectly obvious, that, to subtract a positive quantity, we have only to put the sign — before it; i. e. to change its sign, and add it.

Hence, for subtracting one quantity from another, we have the following

RULE.

§ 60. *Change the signs of the quantity to be subtracted, and then add the two quantities.*

a.) In expressions of this form, $a+(b-c)-(c+y)+(a+b)(c-x)$, the quantities enclosed in the symbol of union (§ 2. *h*), as $(b-c)$, $(c-y)$, and such quantities multiplied together, as $(a+b)(c-x)$, are to be regarded as single *compound*, or *complex* terms; and the rule applies to the sign before the whole term, and not to the signs between the parts of the term. Thus, to subtract $a-(b-c)+(a-b)-(a-b)(-c)$ from g, we write

$$g-a+(b-c)-(a-b)+(a-b)(-c).$$

Here, the addition of $(b-c)$, and the subtraction of $(a-b)$ is *indicated*. If this addition and subtraction also be performed, we shall have $g-a+b-c-a+b+(a-b)(-c)$.

NOTE. An operation is said to be *indicated*, when, without being actually performed, it is denoted by the proper symbol. Thus, $3ab\times 2ab$ is an indicated multiplication. So subtraction is indicated by writing the subtrahend[x] in a parenthesis, and placing the sign — before it. Thus $a-(b-c)$.

b.) In subtracting a term preceded by the double sign, the order of the signs will obviously be inverted. Thus, $a-(\pm b)=a\mp b$; i. e. *plus* or *minus* is changed to *minus* or *plus*. $10-(\pm 5)=10\mp 5=5$, or 15.

1. From a, subtract $b+c$. *Remainder*, $a-b-c$.

2. From $a+b$, subtract c. *Rem.* $a+b-c$.

3. $a+b-(a-b)=$ what? *Ans.* $2b$.

(*x*) Lat. subtrahendus, *to be subtracted*, from subtraho *to take away, subtract.*

4. $\frac{1}{2}x+\frac{1}{2}y-(\frac{1}{2}x-\frac{1}{2}y)=$ what? *Ans.* y. That is, the *half sum* — the *half difference* of any two quantities = the *less*.

5. From x^2-y^2, subtract x^2+xy. *Rem.* $-xy-y^2$.

6. From a^n-b^n, subtract a^n-ba^{n-1}.

7. From $y^2+x^2+2cx+c^2$, subtract $y^2+x^2-2cx+c^2$.

8. From $A^2+2cx+\frac{c^2x^2}{A^2}$, take $A^2-2cx+\frac{c^2x^2}{A^2}$.

9. From a^3-b^3, take a^3-ba^2.

10. From ba^2-b^3, take a^2b-ab^2.

11. From ab^2-b^3, take ab^2-b^3.

12. From $3b^2y^{-1}+4b^{\frac{3}{2}}y^{-\frac{1}{2}}-6by^0-10b^{\frac{1}{2}}y^{\frac{1}{2}}+8b^0y$, take $-2b^2y^{-1}-3by^0+2b^{\frac{3}{2}}y^{-\frac{1}{2}}+5b^{\frac{1}{2}}y^{\frac{1}{2}}-2b^0y+3b^{-\frac{1}{2}}y^{\frac{3}{2}}-3b^{-1}y^2$.

13. From $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$, subtract $-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}$.

14. From $10x-7y=30$, subtract $8x-7y=20$.

15. From $A^2y^2+B^2x^2=A^2B^2$, subtract $A^2y''^2+B^2x''^2=A^2B^2$. *Rem.* $A^2(y^2-y''^2)+B^2(x^2-x''^2)=0$.

§ 61. It is sometimes important to *indicate* the subtraction of a polynomial, without actually performing it.

Thus, $a+x-(b+c-d)$ which, when performed, gives $a+x-b-c+d$.

As, in *performing* a subtraction which has been *indicated*, we change all the signs of the quantity within the parenthesis; so we may return from a performed, to an indicated subtraction, by re-changing all the signs of the quantity whose subtraction is to be indicated, and enclosing the terms in a parenthesis, with the sign — before it. We may, therefore, put a polynomial under different forms, without affecting its value. Thus,

$$b^2-2bc+c^2=b^2-(2bc-c^2)=-(-b^2+2bc-c^2).$$

$$cy''-A^2+B^2=cy''-(A^2-B^2).$$

1. $-(x^2-A^2)=$ what? $R^2-(\cos a \cos b-\sin a \sin b)$?

2. Indicate, in every way possible, without changing the order of the terms, the subtraction of $r-s+t-u$ from a.

§ 62. We have already found, in several instances, two signs combined before a single term (§§ 7. *a*, *b*, 60. *b*). There is nothing to hinder any number of signs from being thus combined. It is proper therefore to consider the effect of such a combination.

a.) In the first place, as addition is simply the bringing of quantities together in their proper character (§ 54), the sign $+$ can never change the previously existing sign of a term. Whether employed once or oftener, it simply leaves the sign of the term as it was before the sign $+$ was prefixed. Thus,

$$a+(-b) \text{ [i. e. } a \text{ together with } -b] = a-b.$$

Hence, in estimating the effect of any number of signs, the positive signs may be disregarded; *the sign of the term depends upon the negative signs.*

b.) As subtraction, on the other hand, always changes the sign of a term, the sign $-$ always reverses the character of the term to which it is prefixed. Thus,

$$+a=a \ (\S\ 4.\ a); \quad \therefore -(+a)=-a.$$

Again $\qquad -(-a)=+a; \qquad$ § 7. *a*, *b*.

$\therefore \qquad -(-(-a)=-(+a)=-a.$

Hence,

§ 63. ***If the number of negative signs before a term be*** EVEN, ***the resulting sign is*** $+$; ***if*** ODD, $-$. Compare § 11. Note 2.

NOTE. This includes the case, in which the signs are all positive. For then the number of negative signs is represented by 0, an even number, being less by unity than 1, which is an odd number.

1. What is the value of $-(b^2-2bc+c^2)$?

Ans. $-(b^2-2bc+c^2)=-(b^2)-(-2bc)-(+c^2)=-b^2+2bc-c^2$.

2. What is the proper sign of $-(-a^2)$? of $-(-(-a^3)$? of $-(-(-(-a^4)$? of $+(+a)$? of $+(+(+a)$? of $+(-a)$? of $+(-(-a)$? of $+(-(+a)$?

PROBLEMS.

§ 64. 1. The remainder found by subtracting $-23-\frac{2}{3}x$ from $2x+7$ is $6x$. What is the value of x?

2. The difference between $8x-5$ and $-7x+12$ is nothing. What is the value of x?

$$8x-5-(-7x+12)=0.$$

3. $\frac{3x}{5}-\frac{7x}{10}-\left(\frac{7x}{8}-\frac{3x}{4}\right)=-15.$ $x=$ what?

4. $Aa-Av-(Bb+Bv)=0.$ $v=$ what?

Ans. $v=\frac{Aa-Bb}{A+B}$.

5. Divide 54 into two such parts, that the less subtracted from the greater, minus the greater subtracted from three times the less, shall be equal to nothing.

Let $x=$ the less;

then $54-x=$ the greater.

$\therefore$ $54-x-x-(3x-(54-x))=0.$

§ 65. 1. A is 10 years older than B, and the sum of their ages is 60. What are their ages?

Let $x=$ A's age;

Then $x-10=$ B's age,

and $x+x-10=$ the sum of their ages, which is 60.

$\therefore$ $x+x-10=60.$

$\therefore$ $2x=70.$ §§ 34, 44.

$\therefore$ $x=35$, A's age,

and $x-10=25$, B's age.

Or, let $x=$ B's age.

Then $x+10=$ A's age, &c.

Or, again, let $x =$ A's age;

and $60-x =$ B's age.

Then $x-(60-x) =$ the difference of their ages, which is 10.

$\therefore \quad x-(60-x) = 10.$

2. The sum of two numbers is 100, and their difference is 20. What are the numbers?

3. The sum of two numbers is S, and their difference is D. What are the numbers?

Ans. The greater is $\frac{S+D}{2}$, or $\frac{1}{2}S+\frac{1}{2}D$, and the less, $\frac{S-D}{2}$, or $\frac{1}{2}S-\frac{1}{2}D$.

NOTE. The 1st, 2d, and 3d examples propose the same question under different forms. But, in the 3d, the quantities employed *remain in the result* (§ 55. 3. N.), and show how they are employed to obtain that result. Thus S denotes the sum of any two numbers, and D, their difference; and we find the greater by adding the difference to the sum, and dividing by two; and the less, by subtracting the difference from the sum, and dividing by two. (Compare § 57. 2, 3, § 60. 3, 4, and Geom. § 22.) Thus,

Let the sum of two numbers be 50, and their difference, 6; and find the numbers. Here $S = 50$, and $D = 6$;

$$\therefore \quad \frac{S+D}{2} = \frac{56}{2} = 28, \text{ and } \frac{S-D}{2} = \frac{44}{2} = 22.$$

And we find $28+22 = 50 = S$, the sum;

and $28-22 = 6 = D$, the difference.

Let the sum be	75,	and difference	25;	what are the numbers?
" "	12,	"	2;	
" "	12,	"	3;	
" "	19,	"	7;	
" "	75°27′,	"	13°15′;	

CHAPTER II.

MULTIPLICATION AND DIVISION.

I. MULTIPLICATION.

§ 66. MULTIPLICATION is the process of *combining factors into a product* (see § 10); in other words, it is the process of taking as a term, one quantity called the *multiplicand*[w], as many times or parts of a time, as there are units or parts of a unit, in another quantity called the *multiplier*.

Thus, if 6 dollars be taken as a term 3 times, the result is $6\times3=6+6+6=18$; if 6 dollars be taken as a term $\frac{2}{3}$ of a time, the result is $6\times\frac{2}{3}=(2+2+2)\frac{2}{3}=2+2=4$.

NOTE. It is obvious, that, in numbers, either factor may be made the multiplicand, and the other, the multiplier, without affecting the result. See Geom. § 172.

MULTIPLICATION OF MONOMIALS.

§ 67. All multiplication resolves itself, as we shall see, into the multiplication of monomials. We shall, therefore, consider that case first.

Numerical coefficients are, of course, subject to the principles of Arithmetic, and must be multiplied accordingly. Letters, we have seen, are multiplied by writing them together (§ 2. *e.* N.); and powers, by adding their exponents

(*w*) Lat. multiplicandus, *to be multiplied*, from multiplico, compounded of multus, *many*, and plico, *to fold*; as if the quantity were folded on, or added to, itself.

(§ 24. a). Hence, we have, for the multiplication of monomials, the following

RULE.

§ 68. *Multiply the numerical coefficients as in Arithmetic; and annex the letters of the factors, giving to each an exponent equal to the sum of its exponents in the factors.*

a.) We have shown (§ 9. a), that the product of *two* factors of *like* signs is *positive*, and of *unlike* signs, *negative*; and (§ 11. N. 2), that the product of *any even* number of *negative* factors is *positive*, and of *any odd* number, *negative*. We have also shown (§ 62. a), that positive signs have no effect to change the sign of a term; but that the sign depends upon the *negative* signs. Hence, whatever be the number of factors,

If the number of *negative* factors be EVEN, the product is *positive*; if ODD, *negative*.

b.) The sign of each factor obviously produces its effect upon the whole product (§ 9. a). Hence, we may write the signs of all the factors before the product, and determine the resulting sign by § 63.

\# c.) When one only of the factors has a *double sign* ($\pm$ or $\mp$), the sign of the product will, of course, be double; and will be either the same as that of the factor, or inverted, according as an *even* or *odd* number of the remaining factors may be *negative*. Thus,

$$\pm a \times b = \pm ab;\ \pm a \times -b = \mp ab;\ \mp ab \times -c = \pm abc.$$

$$(\pm a)(-b)(-c) = \pm abc;\ \pm a . -b . -c . -x = \mp abcx.$$

If two factors have each a double sign, and if it be understood, that the upper signs must be taken together, and the lower signs together, the sign of the product will, obviously, be single; and, if the signs of the factors be *alike*, the product will be *positive*; if *unlike*, *negative*. Thus,

$$\pm a \times \pm b = +b;\ \pm a \times \mp b = -ab.\quad (\pm a)(\pm b)(\mp c) = \mp abc.$$

d.) The degree of the product of several monomial factors is, evidently, equal to the sum of the degrees of those factors (28).

+1. Multiply together $2a^2b$, $-3ab^2$, $4a^{-1}b^{-2}$, and $-\frac{1}{2}b$.

Product $12a^2b^2$.

2. $3a\times-b\times-c\times-2hy=$ what?

3. $a^m\times a^{-n}\times b^n\times ab^{-1}=$ what? *Ans.* $a^{m-n+1}b^{n-1}$.

4. Multiply together $\frac{1}{2}$, $-\frac{1}{3}$, R^{-3}, $-x^2$, and $-x^2$.

5. $(\pm ax)(\pm x)=$ what? $ax^2(\pm x)$? $(-ax^3)(\pm x)$?

MULTIPLICATION OF POLYNOMIALS.

§ 69. First, let one factor only be a polynomial. Thus, Multiply together $b+y$ and a.

$(b+y)$ times a is the same as a times $(b+y)$ [see § 66. N.]; i. e. a times the sum of b and y; which is, obviously, the same as the sum of a times b, and a times y.

$$\therefore \qquad (b+y)a, \text{ or } a(b+y)=ab+ay.$$

Hence, the product of a *polynomial* into a *monomial* consists of the *aggregate* of the products of the monomial into the several *terms* of the polynomial. See Geom. § 178. 1.

1. Multiply $a^2-2ab+b^2$ by a.

Prod. $a^3-2a^2b+ab^2$.

2. $(a^2\pm2ab+b^2)\times\pm b=$ what?

Ans. $\pm a^2b+2ab^2\pm b^3$.

3. $(b^2+c^2-a^2)\times-R^2=$ what?

4. Multiply $1+\frac{1}{2}a^{-2}x^2-\frac{1}{8}a^{-4}x^4+\frac{1}{16}a^{-6}x^6$ by a.

§ 70. Again let there be two polynomial factors. Thus, Multiply $a+b$ by $c+y$.

$(c+y)$ times $a+b$ is evidently the same as c times $a+b$, added to y times $a+b$; i. e.

$(a+b)(c+y)=(a+b)c+(a+b)y=ac+bc+ay+by$. See § 67.

Hence, we have, for the multiplication of polynomials, the following

RULE.

§ 71. *Multiply each term of the multiplicand by each term of the multiplier, and add the products.* See Geom. § 178. Cor. III.

a.) This is precisely the method employed in Arithmetic. Thus, to multiply 34 by 25, we have

$$\begin{array}{rcr} 34 & = & 30+\ 4 \\ 25 & = & 20+\ 5 \\ \hline 170 & = & 150+20 \\ 68 & = & 600+80 \\ \hline 850 & = & 600+230+20 \end{array}$$

1. $(a^2+2b)(a^{-2}-b^2)=$ what? $(a+b)(a+b)=(a+b)^2$?

$$\begin{array}{l} a^2+2b \\ a^{-2}-b^2 \\ \hline a^0+2a^{-2}b \\ \quad -a^2b^2-2b^3 \\ \hline 1+2a^{-2}b-a^2b^2-2b^3 \end{array} \qquad \begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

2. $(a^2+ay+y^2)(a^2-ay+y^2)=$ what?
Ans. $a^4+a^2y^2+y^4$.

3. $(a^2-b^2)(a-b)=$ what? *Ans.* $a^3-a^2b-ab^2+b^3$.

4. $(a+b)(a-b)(a-b)=$ what? $(a+b)(a-b)(a+b)$?

5. $(a^3+3a^2b+3ab^2+b^3)(a^2+2ab+b^2)=$ what?

6. $(2a^4-3a^2b^2+4b^4)(x-y)2x=$ what?

7. $(a+b-c)(a-b+c)=$ what? $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(x^{\frac{1}{3}}-y^{\frac{1}{3}})$?

8. $(x-a)(x+b)=$ what? *Ans.* $x^2\left.\begin{array}{r}-a\\+b\end{array}\right|x-ab$.

9. $(x+a)(x+b)(x-c)(x-e)=$ what?

10. $(az\pm bz^2+cz^3)(1\pm z+z^2\pm z^3)=$ what?
Ans. $az\left.\begin{array}{r}\pm a\\\pm b\\ \\ \end{array}\right|z^2\left.\begin{array}{r}+a\\+b\\+c\end{array}\right|z^3\left.\begin{array}{r}\pm a\\\pm b\\\pm c\end{array}\right|z^4\left.\begin{array}{r}+b\\+c\\ \\ \end{array}\right|z^5\pm cz^6$.

§ 72. If two polynomials are each homogeneous, their product will be homogeneous also. For the degree of any term in the product is equal to the sum of the degrees of a term in each factor (§ 68. *d*); and those degrees being the same throughout, their sum must be always the same, and therefore all the terms of the product will be of the same degree.

Hence, if, in multiplying homogeneous polynomials together, we observe that the degree of one term is greater or less than the degree of the other terms, we may know that some mistake has been made.

This remark is the more important, because so many of the investigations of Algebra, especially those relating to Geometry, give rise to homogeneous expressions.

§ 73. If the product of polynomials contains similar terms, it may, of course, be simplified by § 34. But it is apparent that, if the factors themselves were reduced to their simplest form, there will always be some terms of the product unlike all the others, and, therefore, incapable of any reduction except the partial reduction explained in § 34. *c*. These are,

1. The product of the terms containing *the highest powers of any letter*, in each of the factors; and

2. The product of the terms containing the *lowest powers* of any letter.

For these two terms must contain that letter, the one with a greater, and the other with a less exponent, than any of the other terms or partial products; and, consequently, cannot be similar to any of them. Hence, no product, involving a polynomial factor, can consist of less than two terms.

§ 74. If there are no similar terms in the product of two polynomials, the whole number of terms in the product will be equal to the product of the number of terms in the multiplicand by the number of terms in the multiplier.

For, if there be four terms in the multiplicand, and one

in the multiplier, there will be four terms in the product; another term in the multiplier will give another four terms in the product, and so on.

Also, if we introduce another factor, the same reasoning will apply to the product of this factor into the former product. Hence, in general, if there is no reduction, the number of terms in any product is equal to the continued product of all the numbers of the terms in the several factors.

§ 75. The multiplication of polynomials is frequently *indicated,* without being performed. Thus,

$$a(y+h)^2=a(y^2+2yh+h^2)=ay^2+2ayh+ah^2.$$

$$(p-b)\times(p-c);\ \overline{a+b-c}\,.\,\overline{a+c-b};\ \tfrac{1}{2}s(\tfrac{1}{2}s-a).$$

When a multiplication, so indicated, is performed, the expression is sometimes said to be *developed.*

MULTIPLICATION BY DETACHED COEFFICIENTS.

§ 76. In multiplying polynomials arranged according to the powers of any common letter or letters, that letter or those letters may be omitted in the operation, and the powers supplied in the result; the product of the highest or lowest powers being placed in the first term, and the powers then regularly descending or ascending through all the terms.

This is called multiplication by DETACHED COEFFICIENTS; and will be best explained by a few examples. Thus,

To multiply x^2+2x+1 by x^2-2x+1, we write the coefficients, and multiply, as follows:

$$\begin{array}{l}
1+2+1 \\
\underline{1-2+1} \\
1+2+1 \\
\quad -2-4-2 \\
\qquad\quad \underline{1+2+1} \\
1+0-2+0+1.
\end{array}$$

Supplying the powers of x, we have $x^4+0x^3-2x^2+0x+1=x^4-2x^2+1$.

Multiply $a^2+2ab+b^2$ by $a+b$.

Here the polynomial factors being arranged with respect to both the letters, both may be omitted, and afterwards supplied, one with descending, the other with ascending powers. Thus,

$$\begin{array}{l} 1+2+1 \\ \underline{1+1} \\ 1+2+1 \\ \underline{1+2+1} \\ 1+3+3+1. \end{array}$$

Supplying the letters, we have $a^3+3a^2b+3ab^2+b^3$.

a.) In adding the coefficients of the partial products in the first example, we obtain zero in the second and fourth places. The cypher must be written, to occupy the place of the term, and show what powers of the letters fall out. In like manner, if any power of a letter, between the highest and lowest in any factor, be wanting, zero should be regarded as its coefficient, and written in its place. This will fill out the series, and will, obviously, cause the coefficients of similar terms to stand under one another. Thus,

3. Multiply $a^2+2ay+y^2$ by a^2-y^2.

$$\begin{array}{l} 1+2+1 \\ \underline{1+0-1} \\ 1+2+1 \\ 0+0+0 \\ \underline{-1-2-1} \\ 1+2+0-2-1. \end{array}$$

$\therefore$ The product is $a^4+2a^3y-2ay^3-y^4$.

4. Multiply $z^3-3z^2y+3zy^2-y^3$ by $z^2-2zy+y^2$.

5. $(a+b)^3=$ what? $(a+b)^4$? $(a+b)^5$?

6. $(z^3+z^2y+zy^2+y^3)(z-y)=$ what?

PROBLEMS.

§ 77. 1. Given $x-\frac{1}{3}(2x+1)=\frac{1}{4}(x+3)$ to find x.

Ans. $x=13$.

2. Given $\frac{3x+4}{5}+2x=\frac{23-x}{5}+16$, to find x.

3. Given $16x+5=\frac{(4x+14)(36x+10)}{9x+31}$, to find x.

Ans. $x=5$.

4. Given $(ac+br)^2+b^2x=2abcr+(a^2+b^2)c^2$, to find x.

Ans. $x=c^2-r^2$.

6. Given $x^2+x^{-2}=(x-x^{-1})^2+x$, to find x (§ 49).

Ans. $x=2$.

7. A's age is to B's as 2 to 3; and if they live 15 years, A's age will be $\frac{3}{4}$ of B's. What are their ages?

Let $x=$ B's age;

then $\frac{2}{3}x=$ A's age.

Moreover $x+15$, and $\frac{2}{3}x+15$ will be their ages after 15 years.

$$\therefore \qquad \tfrac{2}{3}x+15=\tfrac{3}{4}(x+15).$$

Ans. A's age, 30; B's, 45.

8. A's age is $\frac{1}{2}$ of B's; and 18 years ago, A's age was $\frac{1}{3}$ of B's. What are their ages?

II. DIVISION.

§ 78. *Division* is the process, by which, *having a product and one of its factors, we find the other factor* (see § 10); in other words, it is the process of finding *how many times, or parts of a time, one quantity is contained in another.*

Thus, if 12 be a product, and 3 be one of its factors, the other factor is 4; or 3 is contained in 12, 4 times; if 12 be a product, and 24 be one of the factors, the other factor is $\frac{1}{2}$; or 24 is contained in 12, $\frac{1}{2}$ a time.

DIVISION OF MONOMIALS.

§ 79. As in multiplication, so in division, whatever be the quantities involved, the operation is actually performed upon monomials only. We shall, therefore, consider first the division of monomials.

Numerical coefficients are, of course, subject to the principles of Arithmetic, and must be divided accordingly. Letters, we have seen, are divided by suppressing in the dividend the letters of the divisor (§ 10. *b*); i. e. by subtracting the exponents of the letters in the divisor from the exponents of the same letters in the dividend (§§ 16, 24. *b*). See also § 13. Hence, we have, for the division of monomials, the following

RULE.

§ 80. *Divide numerical coefficients as in Arithmetic; and annex all the literal factors, which remain after suppressing in the dividend those of the divisor.*

a.) If the exponent of any letter be *greater* in the dividend, than in the divisor, its exponent in the quotient will be *positive;*

If *equal*, it will be *zero;* i. e. the letter will disappear; and

If *less*, it will be *negative.*

Or, in the last case, the division may be expressed, as we have seen, by placing the letters, with positive exponents, as the denominator of a fraction, of which the remaining factors of the dividend constitute the numerator (§ 10. *c*).

b.) In case of a single division, we have shown, that, as in multiplication, *like* signs give $+$, *unlike*, $-$. In case of successive division by several divisors, the same rule, of course, applies to each operation. Or, bringing the signs together, as in subtraction (§ 63) and multiplication (§ 68. *a*) we may regard only the *negative* signs. If the number of these be *even*, the quotient is *positive*; if *odd*, *negative.*

c.) The law of the signs may be otherwise demonstrated, as follows. To divide by any quantity is the same as to multiply by its reciprocal (§ 19. Cor. IV.); and the reciprocal of a quantity evidently has the same sign as the quantity itself (§ 18). Therefore, to *divide* by any number of divisors is the same as to *multiply* by the same number of multipliers having each the same sign. Hence, the law of the signs is *the same in division as in multiplication.*

d.) One quantity is commonly said to be *divisible* by another, when the division does not give rise to fractional coefficients, or to negative exponents.

NOTE. *Any* quantity may be said to be divisible by *any* other. For, whatever be the dividend and given factor, another factor can always be found, which will produce the dividend. It is, however, convenient, in many cases, to distinguish as *perfect* or *exact*, the divisibility above mentioned which does not give rise to fractional expressions.

1. $20a^5b^3c \div 4a^3b^3c^3 =$ what? *Ans.* $5a^2c^{-2}$.

2. $a^2b^{-2} \div a^{-1}b =$ what? $a^{\frac{1}{2}}x^2 \div a^{\frac{1}{4}}x^{-2}y$? $-a^{\frac{1}{2}} \div a^{-\frac{1}{2}}$?

3. $a^mb^{-m} \div a^nb^{-n} =$ what? $(a+x)^{\frac{3}{2}} \div (a+x)^{-\frac{1}{2}}$?

TO DIVIDE A POLYNOMIAL BY A MONOMIAL.

§ 81. In multiplying a polynomial by a monomial, we multiply each term of the polynomial by the monomial, and add the products (§ 69). Therefore, reversing the process, we have, for dividing a polynomial by a monomial, the following

RULE.

Divide each term of the dividend by the divisor, and add the quotients.

Thus, $(ab \pm ay) \div a = b \pm y$.

1. Divide x^2y+xy^2 by xy. *Quotient,* $x+y$.

2. $(ax \pm x^2) \div x =$ what? $(rs - s) \div s$?

3. $(2rx - x^2) \div x =$ what? $(A^2B^2 - B^2x^2) \div B^2$?

4. $(-R \cos b \cos c + R \sin b \sin c) \div -R =$ what?

5. $(a - x) \div a =$ what? *Ans.* $1 - a^{-1}x$, or $1 - \frac{x}{a}$.

6. Divide $R^2 - x^2$ by R^2.

7. " $R - \frac{1}{2}R^{-1}x^2 - \frac{1}{8}R^{-3}x^4$ by R.

8. " $a^{-\frac{2}{3}} - \frac{2}{3}a^{-\frac{8}{3}}x^2 + \frac{5}{9}a^{-\frac{14}{3}}x^4$ by $a^{-\frac{2}{3}}$.

TO DIVIDE ONE POLYNOMIAL BY ANOTHER.

§ 82. Divide $3ab^2 + 3a^2b + a^3 + b^3$ by $a + b$.

a.) This dividend being regarded as the product of the divisor and quotient (§ 10), the terms containing the highest and the lowest powers of a and b must consist of the unreduced products of the highest and of the lowest powers of those letters in the two factors (§ 73. 1, 2).

b.) If, therefore, we divide the term of the dividend which contains the highest power of a, by the term of the divisor which contains the highest power of the same letter, we must obtain the corresponding term of the quotient.

c.) If, now, we multiply the divisor by the term of the quotient, which we have found, we shall have one of the partial products whose sum is the dividend.

d.) If, then, we subtract this partial product, there will remain the sum of the other partial products, viz. of the divisor into the other terms of the quotient.

e.) There will, of course, be a highest power of a in this new or remaining dividend, which term divided by the term containing the highest power of a in the divisor, as before, will give a term containing the highest power of a in the remaining terms of the quotient; and so on.

f.) And, as the sum of the products of all the terms of the divisor by each term of the quotient must make up the

dividend, if we subtract those partial products, one after another from the dividend, they must exhaust it; and the remainder, after the last subtraction, will be zero.

g.) If we obtain a remainder equal to zero by simply dividing the first term of each remainder by the first term of the divisor, the division is said to be *exact*, and the dividend is said to be *divisible* by the divisor (§ 80. *d*, N.).

h.) If, however, after exhausting the given terms of the dividend, we still have a remainder, the division may be immediately completed by writing the whole remainder over the whole divisor, for the last term of the quotient; or the division may be still farther continued (§ 87) according to the rule, and terminated, whenever we please, by a fractional term, as above indicated.

i.) These operations will be more conveniently performed, if the dividend and divisor be first arranged with respect to the powers of some one letter (§ 33. *a*).

This arrangement may be according to either the ascending or the descending powers of the letter. The descending order, however, is most commonly employed.

k.) The polynomials above being arranged with reference to a, and placed in order for division, will stand thus; the divisor being placed at the right of the dividend, and the quotient under the divisor.

$$\begin{array}{r|l}
a^3+3a^2b+3ab^2+b^3 & a+b \\ \cline{2-2}
\underline{a^3+\ \ a^2b\qquad\qquad\quad} & a^2+2ab+b^2 \\
2a^2b+3ab^2+b^3 & \\
\underline{2a^2b+2ab^2\quad} & \\
ab^2+b^3 & \\
\underline{ab^2+b^3} & \\
0 &
\end{array}$$

From the reasoning above, we deduce the following general

RULE.

§ 83. 1. *Arrange both dividend and divisor according to the powers of some common letter, either ascending, or descending in both.*

2. *Divide the first term of the dividend by the first term of the divisor* (§ 80), *and set the result, with its proper sign, as a term of the quotient.*

3. *Multiply the divisor by this first term of the quotient, and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor, set the result in the quotient with its proper sign, multiply, and subtract as before, and continue the process as long as the case may require.*

1. Divide $a^3+3a^2x+x^3+3ax^2$ by $a+x$.

2. Divide $x^4+6y^2x^2+4x^3y+4xy^3+y^4$ by $x^2+2xy+y^2$.

a.) It is not necessary to write all the remaining terms of the dividend, after each subtraction. Indeed, none *need* be written, except those which change their form by subtraction and reduction. It is convenient, however, to bring down one additional term of the dividend, at each subtraction. This is the method commonly practised.

1. Divide $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$ by $a-x$.

2. Divide $a^4+a^2z^2+z^4$ by a^2+az+z^2.

3. Divide $x^5+6x^4-10x^3-112x^2-207x-110$ by $x^2+7x+10$. *Quot.* $x^3-x^2-13x-11$.

4. Divide $a^6-3a^4x^2+3a^2x^4-x^6$ by a^2-x^2.

5. Divide $1-a^{-1}b$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$. *Quot.* $a^{-\frac{1}{2}}+a^{-1}b^{\frac{1}{2}}$.

6. Divide $\frac{1}{12}a^3-\frac{1}{8}a^2b-\frac{1}{15}ab^2+\frac{1}{10}b^3$ by $\frac{1}{3}a-\frac{1}{2}b$.
Quot. $\frac{1}{4}a^2-\frac{1}{5}b^2$.

7. Divide $x^3+ax^2-bx^2+cx^2-abx+acx-bcx-abc$ by $x^2+ax-bx-ab$.

$$\begin{array}{lr|r|r||r|l}
 & x^3+a & x^2-ab & x-abc & x^2+a & x-ab \\
 & -b & +ac & & -b & \\
\cline{5-5}
 & +c & -bc & & \multicolumn{2}{l}{x+c \quad \textit{Quotient.}} \\
 & x^3+a & x^2-abx & & & \\
 & -b & & & & \\
\cline{2-4}
\textit{1st Rem.} & & cx^2+ac & x-abc & & \\
 & & -bc & & & \\
 & & cx^2+ac & x-abc & & \\
 & & -bc & & & \\
\cline{3-4}
\end{array}$$

b. The operation may be still further shortened. Arrange, divide and multiply, as directed; but, instead of writing the product under the dividend, subtract each term mentally, as it is formed, and write the reduced remainder (§ 83. *a*). Thus,

$$\begin{array}{ll|l}
 & a^4-4a^3x+6a^2x^2-4ax^3+x^4 & a^2-2ax+x^2 \\
\cline{3-3}
\textit{1st Rem.} & -2a^3x+5a^2x^2-4ax^3 & a^2-2ax+x^2 \\
\textit{2d Rem.} & a^2x^2-2ax^3+x^4 & \\
\end{array}$$

1. Divide $x^2-7x+12$ by $x-3$.

2. Divide $2a^{2m}+2a^mb^p-4a^mc^n-3a^mb-3b^{p+1}+6bc^n$ by $2a^m-3b$. *Quot.* $a^m+b^p-2c^n$.

§ 84. *c.*) We need only the *first* term of each remainder (§ 83. 4). The other terms are simply reserved till we subtract from them the terms of the next product, and so on. Instead, therefore, of performing these successive subtractions, we may write the similar terms of the several products under one another, and subtract the aggregate of each set, when the corresponding first term of a remainder is required for division.

Or we may change the signs of the several terms of the products as we write them, and *add* each column as we come to it. If we adopt this course, we shall be less liable to mistake, if we change the signs of the *divisor*, all except the first, which should remain unchanged, to prevent mistakes in the signs of the quotient; and which can occasion

no mistake in subtracting, as its product always cancels the term above it, and need not be written.

Divide $a^4-4a^3x+6a^2x^2-4ax^3+x^4$ by $a^2-2ax+x^2$.

$$
\begin{array}{l|l}
a^4-4a^3x+6a^2x^2-4ax^3+x^4 & \underline{a^2+2ax-x^2} \\
\underline{+2a^3x-a^2x^2} & a^2-2ax+x^2 \\
-2a^3x & \\
\qquad\underline{-4a^2x^2+2ax^3} & \\
\qquad+a^2x^2 & \\
\qquad\qquad\underline{+2ax^3-x^4} &
\end{array}
$$

d.) The last method is conveniently written as follows. Write the terms of the divisor under one another, on the left of the dividend, changing the signs of all but the first. Write the terms of the partial products, except the first of each, diagonally under the corresponding terms of the dividend. Below, in a horizontal line, write the first terms of the remainders as they are formed, each under the column from which it is produced. Write the quotient also in a horizontal line below the last, each term under the term of the dividend, from which it was formed. Thus,

$$
\begin{array}{r|l}
a^2 & a^4-4a^3x+6a^2x^2-4ax^3+x^4 \\
+2ax & \quad +2a^3x-4a^2x^2+2ax^3 \\
-x^2 & \qquad\qquad -a^2x^2+2ax^3-x^4 \\
\hline
 & \quad -2a^3x+a^2x^2 \\
\hline
\end{array}
$$

Quotient, $a^2-2\,ax+x^2$.

NOTES. (1.) If any term in the series of powers be wanting, its place should be filled with a cypher (§ 76. *a*); or the given terms should be placed at such distances from each other, that like terms of the partial products may stand under them. (2.) Each term of the partial products will stand against that term of the divisor from which it is formed.

1. Divide $a^6+2a^3z^3+z^6$ by a^2-az+z^2.
2. Divide $a^3+a^2b-ab^2-b^3$ by $a-b$.

DIVISION BY DETACHED COEFFICIENTS.

§ 85. Division, as well as multiplication, may be performed by DETACHED COEFFICIENTS. Thus,

1. Divide $a^3-3a^2b+3ab^2+b^3$ by $a-b$.

$$\begin{array}{l|l} 1-3+3-1 & 1-1 \\ \cline{2-2} \underline{1-1} & 1-2+1 \\ \quad -2 & \\ \quad \underline{-2+2} & \\ \qquad 1 & \\ \qquad \underline{1-1} & \end{array}$$

Supplying the letters, by dividing the first term of the dividend by the first term of the divisor, we have $a^2-2ab+b^2$.

2. Divide a^4-b^4 by a^2-b^2.

$$\begin{array}{l|l} 1+0+0+0-1 & 1+0-1 \\ \cline{2-2} \underline{1+0-1} & 1+0+1. \therefore \ Quot. = a^2+b^2 \\ \quad 0+1 & \\ \qquad \underline{1+0-1} & \end{array}$$

SYNTHETIC DIVISION.

§ 86. SYNTHETIC[x] DIVISION is division with detached coefficients, performed by the method of § 84. *d*. With detached coefficients, however, the method admits of simplification, when the first coefficient of the divisor is 1. For, in this case, the coefficient of each term in the quotient will be the same as the corresponding coefficient of the first term of the dividend or remainder; and may, therefore be found by simply adding the coefficients above it. Thus, to divide $a^4-4a^3x+6a^2x^2-4ax^3+x^4$ by $a^2-2ax+x^2$.

$$\begin{array}{r|l} 1 & 1-4+6-4+1 \\ +2 & \ \ +2-4+2 \\ -1 & \qquad -1+2-1 \\ \cline{2-2} & 1-2+1+0+0 \ \therefore \ Quot. = a^2-2ax+x^2. \end{array}$$

Moreover, if the first coefficient of the divisor be not 1, it can evidently be made so, by dividing both divisor and dividend by the given first coefficient.

(*x*) Gr. σύνθεσις, *composition, putting together;* each term of the quotient being formed by *adding* the like terms of the dividend and of the partial products with their signs changed.

1. Divide x^3-3x^2+3x-1 by x^2-2x+1.

2. Divide $4a^4-9a^2b^2+6ab^3-b^4$ by $2a^2-3ab+b^2$.

Solve the examples of §§ 83, 84 by this method, observing, when the series of powers is not complete, to fill the place of the missing terms with cyphers (76. *a*).

INFINITE SERIES.

§ 87. When an exact division is impossible, the operation may still be carried to any extent, forming what is called an *infinite*(y) *series.* The process is similar to the process of approximation in the division of decimals in arithmetic.

Thus, to divide a by $a+x$.

$$\begin{array}{l|l}
a & a+x \\ \cline{2-2}
\underline{a+x} & 1-a^{-1}x+a^{-2}x^2-a^{-3}x^3+a^{-4}x^4-\&c. \\
\quad -x & \\
\quad \underline{-x-a^{-1}x^2} & \\
\qquad\quad a^{-1}x^2 & \\
\qquad\quad \underline{a^{-1}x^2+a^{-2}x^3} & \\
\qquad\qquad\quad -a^{-2}x^3 & \\
\qquad\qquad\quad \underline{-a^{-2}x^3-a^{-3}x^4} & \\
\qquad\qquad\qquad\qquad a^{-3}x^4 &
\end{array}$$

Or (§ 86), thus,

$$\begin{array}{r|l}
1 & 1 \\
-1 & \;-1+1-1+1-\&c. \\ \hline
 & 1-1+1-1+1-\&c. \;\therefore\; Quot. = 1-a^{-1}x+a^{-2}x^2-\&c
\end{array}$$

We have, therefore,

$$\frac{a}{a+x}=1-a^{-1}x+a^{-2}x^2-a^{-3}x^3+a^{-4}x^4-a^{-5}x^5+\&c.(1.)$$

or, in another form (§ 14),

$$\frac{a}{a+x}=1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^4}-\frac{x^5}{a^5}+\frac{x^6}{a^6}-\frac{x^7}{a^7}+\&c. \ (2.)$$

a.) We find here a series of terms, alternately positive and negative, beginning with unity or the zero power of

(y) Lat. infinitus, *without end.*

both x and a, and containing in the successive terms the powers of x increasing, and those of a decreasing by unity, without limit, that is, *infinitely;* the numerical coefficient of each term being unity. As soon as we have discovered this order, which is called the *law* of the series, we may write the terms to any extent, without the labor of dividing.

b.) 1. Let $a=10$ and $x=1$; then

$$\frac{a}{a+x}=\frac{10}{10+1}=\frac{10}{11}=1-\frac{1}{10}+\frac{1}{100}-\frac{1}{1000}+\frac{1}{10000}-\&c.$$
$$=1+\frac{1}{100}+\frac{1}{10000}+\&c.-\left(\frac{1}{10}+\frac{1}{1000}+\&c.\right)=$$
$$1.010101\&c.-(.1010101\&c.)=.9090909\&c.$$

2. Let $a=1$, $x=10$; then

$$\frac{a}{a+x}=\frac{1}{11}=1-10+100-1000+10{,}000-\frac{100{,}000}{11}=$$
$$(1+100+10{,}000)-\left(10+1000+\frac{100{,}000}{11}\right)=$$
$$10{,}101-10{,}100\tfrac{10}{11}=\frac{1}{11}.$$

3. Let $a=100$ and $x=1$; then $\frac{a}{a+x}\left(=\frac{100}{101}\right)=$ what?

4. Let $a=1000$, and $x=1$; then $\frac{a}{a+x}=$ what?

c.) 1. Develop $(a+x)^{-1}\left(=\frac{1}{a+x}\right)$.

Ans. $\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\&c.$ or $\frac{1}{a}\left(1-\frac{x}{a}+\frac{x^2}{a^2}-\&c.\right)$

Let $x=1$, $a=10$, 100, &c.

2. Develop $(a-x)^{-1}$. *Ans.* $\frac{1}{a}\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\&c.\right)$.

Substitute for a and x as above.

3. Develop $b(a-x)^{-1}\left(=\frac{b}{a-x}\right)$.

Ans. $\frac{b}{a}\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+\frac{x^5}{a^5}+\&c.\right)$.

4. Develop $\frac{1}{1+u^2}$. *Ans.* $1-u^2+u^4-u^6+$&c.

Let $u=\frac{1}{2}, \frac{1}{3}, \frac{1}{10}$, &c.

5. Develop $(a+x)^{-2}\left(=\frac{1}{(a+x)^2}=\frac{1}{a^2+2ax+x^2}\right)$.

Ans. $\frac{1}{a^2}\left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\text{\&c.}\right)$.

d.) Series, in which the terms become less and less as we proceed, as in *b.* 1, 3, and 4, above, are called *converging* series, and are of great utility in the higher applications of Algebra. When the terms continually increase, as in *b.* 2, above, the series is called *diverging.*

Converging series may be treated precisely as approximating decimals in Arithmetic; viz. a few terms may be taken for the whole series, the remaining terms being so small, that they may be neglected without sensible error. Thus, in reducing $\frac{1}{3}$ to a decimal by the common process, we obtain $\frac{1}{3}=.33333333$ &c. But, we may apply the formula in *c.* 3, above, by making $b=3$, $a=10$, and $x=1$; then $\frac{b}{a-x}=\frac{3}{10-1}=\frac{3}{9}=\frac{1}{3}$. Making the substitution, we shall have,

$$\frac{1}{3}=\frac{3}{10-1}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\text{\&c.}=.3333\text{\&c.}$$

e.) In *c.* 1, 2, 3, and 5, the series will converge, whenever $x<a$; when x is $>a$, they will diverge. The series of *c.* 4, will converge, when $u<1$.

f.) In a *converging* series, the remainder, after a few terms, may be neglected; in a *diverging* series, the remainder must always be taken into the account, and constitutes a most important part of the result.

THEOREMS.

§ 88. Algebra employs *general* symbols of quantity (§ 1). Its *results*, therefore, are *general* (§§ 7. *a*, *b*, 55. N., 57. 3, 60.

4, and 65. N.); and whatever is proved of numbers represented by algebraic symbols, is, of course, *demonstrated*[x] of all numbers whatever. *General truths or principles thus demonstrated,* are called THEOREMS[y].

§ 89. Thus, if we square $a+b$, we have

$(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2$. That is,

THEOREM I. *The square of the sum of two numbers is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Or, more briefly,

The square of the sum of two numbers is equal to the sum of their squares, PLUS *twice their product.* See Geom. § 180. Cor. v.

1. $(a+x)^2=$ what? $(a^2+x^2)^2$? $(x+2a)^2$? $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$?

2. $(ab+bc)^2=$ what? $(x+\frac{1}{2}p)^2$? $(1+2mn)^2$?

So in Arithmetic; $(16)^2=(10+6)^2=10^2+2.10.6+6^2$ $=100+120+36=256$.

$(75)^2=(70+5)^2=$ what? $(93)^2$? $(11)^2$? $(19)^2$? $7^2=(4+3)^2$? $(112)^2=(100+12)^2$?

NOTE. An absolute equation which expresses a general result or a theorem, is called a FORMULA[z].

§ 90. If we put $-b$ for b, and apply the principle of § 89, we have,

$(a+(-b))^2=(a-b)^2=a^2+2a(-b)+(-b)^2=a^2-2ab$ $+b^2$ [§ 11. N. 2.]. Hence,

THEOR. II. *The square of the difference of two numbers is equal to the sum of their squares,* MINUS *twice their product.* See Geom. § 183. Cor. VII.

Multiply $a-b$ by $a-b$, and see if the same formula results.

1. $(x-\frac{1}{2}p)^2=$ what? $(x-x'')^2$? $(\sqrt{a}-\sqrt{b})^2$? See § 23. *d.* $(x' \sin a - y' \sin a')^2$? See § 92. N.

(x) Lat. demonstro, *to show, prove beyond the possibility of doubt.* (y) Gr. θεώρημα, from θεωρέω, *to view, contemplate.* (z) Lat., *form, model, rule.*

2. $(-A+x)^2 =$ what? $(2a^2-6x^2)^2$? $(1-a^{-\frac{1}{2}}x^{\frac{1}{2}})^2$?

Note. We have evidently $(a-b)^2 = (b-a)^2$. So $(10-1)^2 = (1-10)^2$; or $9^2 = (-9)^2$ See § 11. N. 2.

In like manner in Arithmetic; $9^2 = (10-1)^2 = 10^2 - 2.10.1 + 1^2 = 100 - 20 + 1 = 81$.

$(98)^2 = (100-2)^2 =$ what? $(75^2) = (80-5)^2$? $(47)^2 = (50-3)^2$? $(93)^2 = (100-7)^2 = (90+3)^2$?

§ 91. *a.*) $(a+b)^2 = a^2+2ab+b^2$. § 89.

And $(a-b)^2 = a^2-2ab+b^2$. § 90.

∴ Adding and subtracting the equations

$(a+b)^2+(a-b)^2 = 2a^2+2b^2 = 2(a^2+b^2)$. Geom. § 199.

$(a+b)^2-(a-b)^2 = 4ab$. Geom. § 184. Hence,

Cor. (1.) *The square of the sum,* PLUS *the square of the difference, of two numbers, is equal to twice the sum of their squares.* (2.) *The square of the sum,* MINUS *the square of the difference, of two numbers, is equal to four times their product.*

§ 92. Multiply $a+b$ by $a-b$.

We have, $(a+b)(a-b) = a^2-b^2$. Hence,

Theor. III. *The product of the sum and difference of two numbers is equal to the difference of their squares.* See Geom. § 185. Cor. IX.

1. $(x+\frac{p}{2})(x-\frac{p}{2}) =$ what? $(A+x)(A-x)$? $(y'+y'')(y'-y'')$? $((1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}})((1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}$?

2. $(R+x)(R-x) =$ what? $(AB+BC)(AB-BC)$? $(x^2+y^2)(x^2-y^2)$? $(x^3+y^3)(x^3-y^3)$?

3. $(\sin a \cos b+\sin b \cos a)(\sin a \cos b-\sin b \cos a) =$ what? *Ans.* $\sin^2 a \cos^2 b - \sin^2 b \cos^2 a$.

Note. $\sin^2 a$ denotes the *square of the sine of a*. This notation is more precise than $\sin a^2$, which might mean the *sine of the square* of a; and is less cumbrous than $(\sin a)^2$. The same remark applies to $\cos^2 a$, $\tan^2 a$, &c.

4. $(a+b+c)(a+b-c)$ [i. e. the sum of $a+b$ and c, into the difference of $a+b$ and c] $=$ what?

Ans. $(a+b)^2-c^2=a^2+2ab+b^2-c^2$.

5. $(a+b-c)(a-b+c)(=(a+\overline{b-c})(a-\overline{b-c}))=$ what?

So in Arithmetic; $12\times8=(10+2)(10-2)=10^2-2^2$ $=100-4=96$.

$19\times21(=(20-1)(20+1))=$ what? 103×97? 51×49? 101×99? 1004×996? $1000\frac{1}{2}\times999\frac{1}{2}$?

§ 93. The same formulæ, read with the second member first, give the *converse*[a] of the above theorems; and enable us to resolve several classes of polynomials into their factors (§ 75). Thus,

(I.) *The sum of the squares,* PLUS *twice the product, of two numbers, is equal to the square of their sum.*

1. $x^2+2xy+y^2=$ what? *Ans.* $(x+y)^2$.

2. $y^2+2yy''+y''^2=$ what? $1+2n+n^2$? $9a^4+24a^2b^2+16b^4$? $169\ (=100+2.10.3+9=10^2+2.10.3+3^2)$?

(II.) *The sum of the squares,* MINUS *twice the product, of two numbers, is equal to the square of their difference.*

1. $x^2-px+\frac{1}{4}p^2=$ what? *Ans.* $(x-\frac{1}{2}p)^2$.

2. $b^2-2bc+c^2=$ what? $1-4n+4n^2$? $a^2-12ab+36b^2$? $81(=100-2.10.1+1=10^2-2.10.1+1^2)$?

(III.) *The difference of two squares is equal to the product of the sum and difference of their roots* (§ 23).

1. $R^2-x^2=$ what? *Ans.* $(R+x)(R-x)$.

2. $\sin^2 a-\sin^2 b=$ what? x^4-y^4? $(AB)^2-(BC)^2$? a^6-b^6? $a^2-b^2+2bc-c^2(=a^2-(b-c)^2$ [§§ 63. 1, 90.])? $b^2+2bc+c^2-a^2$? $1-\cos^2 v(=1^2-\cos^2 v)$? $x^2-x''^2$?

§ 94. 1. Divide a^2-b^2 by $a-b$. *Quot.* $a+b$.

(*a*) Lat. conversus, *turned about.* Of two propositions or sentences, each is said to be the *converse* of the other, when the *condition* of the first is the *conclusion* of the second, and the *conclusion* of the first is the *condition* of the second; or when, in like manner, subject and predicate change places. See Geom. § 32. Note *n*.

2. Divide a^3-b^3 by $a-b$. *Quot.* a^2+ab+b^2.
3. " a^4-b^4 by $a-b$. " $a^3+a^2b+ab^2+b^3$.
4. " a^n-b^n by $a-b$.

Employ the method of § 84. d; thus[b],

$$\begin{array}{r|l} a & a^n \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; -b^n. \\ +b & \quad a^{n-1}b+a^{n-2}b^2+\ldots\ldots+a^2b^{n-2}+ab^{n-1}+b^n. \\ \hline \end{array}$$
$$Quot.=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots\ldots+ab^{n-2}+b^{n-1}.$$

Now, in the successive terms of the remainders, the exponents of a diminish, and those of b increase by unity. The terms of the quotient, of course, follow the same law; and the sum of the exponents in each term of the quotient is $n-1$. Hence we shall find a term containing a^0 and b^{n-1}. This term, multiplied by b, will give b^n, which added, will cancel $-b^n$ in the dividend, and leave a remainder equal to zero, showing a perfect division (§ 82. g).

§ 95. *a.*) Otherwise,

$$\begin{array}{l|l} a^n-b^n & a-b \\ \cline{2-2} a^n-a^{n-1}b & a^{n-1}+\&c. \\ \hline a^{n-1}b-b^n=(a^{n-1}-b^{n-1})b. & \end{array}$$

Now, if $a-b$ will divide (§ 82. g) this remainder, it will, obviously, divide the whole dividend. But it will, evidently, divide this remainder, if it will divide one of its factors, $a^{n-1}-b^{n-1}$. Hence, if $a^{n-1}-b^{n-1}$ is divisible by $a-b$, a^n-b^n is also divisible by $a-b$. That is, if the difference of like powers is divisible by the difference of their roots, the difference of the powers of the *next higher degree* is divisible in like manner.

But we have found a^4-b^4 divisible by $a-b$; $\therefore a^5-b^5$ is divisible by $a-b$; so a^6-b^6, a^7-b^7; and so on, without limit. Or, a^1-b^1 is divisible by $a-b$; $\therefore a^2-b^2$ is divisible by $a-b$; and so on. Hence (§§ 94, 95),

(b) Here, the sign of b being changed, the second term of each product is, without any reduction, the first term of the corresponding remainder, and, of course, need be written but once. We cannot write all the terms of the quotient, unless we assign a particular value to n. The whole number of terms is, obviously, equal to n.

§ 96. THEOR. IV. *The difference of any two positive integral powers of the same degree is divisible by the difference of their roots.*

NOTES. (1.) This method of proof is of great utility in Algebra, and should be perfectly understood. It consists in showing, that, *in how many soever instances a principle has been found true, it will be true in one instance more.* If it be true in $n-1$ cases, it will be in n; if in n, then in $n+1$; if it be true in one instance, it will be true in the second; if in the second, then in the third, and so on. (2.) The limitation to "*positive integral powers*" is necessary; for the principle has been applied to such powers only. And, if $n-1$ is a positive integer, n, obviously, cannot be either negative or fractional.

a.) $\dfrac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+\ldots\ldots+ab^{n-2}+b^{n-1}$.

b.) If $a=b$, we have $\dfrac{a^n-b^n}{a-a}=na^{n-1}$; a result which will be considered hereafter.

c.) $a^{-n}-b^{-n}=(a^{-1})^n-(b^{-1})^n$. § 24. *d.*

$\therefore$ $a^{-n}-b^{-n}$ is divisible by $a^{-1}-b^{-1}$. § 96. Or, which is the same thing, $\dfrac{1}{a^n}-\dfrac{1}{b^n}$ is divisible by $\dfrac{1}{a}-\dfrac{1}{b}$.

d.) $a^{\pm\frac{n}{m}}-b^{\pm\frac{n}{m}}$ $(=(a^{\pm\frac{1}{m}})^n-(b^{\pm\frac{1}{m}})^n)$ is, evidently, divisible by $a^{\pm\frac{1}{m}}-b^{\pm\frac{1}{m}}$.

§ 97. *e.*) $a^{2n}-b^{2n}(=(a^2)^n-(b^2)^n)$ is divisible by a^2-b^2 (§ 96), i. e. by $(a+b)(a-b)$.

$\therefore$ $a^{2n}-b^{2n}$ is divisible by $a+b$.

Now $2n$, being divisible by 2, is an even number. Hence,

Cor. I. *The difference of any two* EVEN *positive integral powers is divisible by the* SUM *of their roots.*

To divide $a^{2n}-b^{2n}$ by $a+b$, employ the method of § 86.

$$\begin{array}{r|l}
1 & 1\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad-1 \\
-1 & -1+1-1+\ldots\ldots\ldots\ldots-1+1-1+1 \\
\hline
 & 1-1+1-1+\ldots\ldots\ldots\ldots-1+1-1+0
\end{array}$$

$\therefore (a^{2n}-b^{2n})\div(a+b)=a^{2n-1}-a^{2n-2}b+a^{2n-3}b^2-\ldots$
$\ldots\ldots+a^3b^{2n-4}-a^2b^{2n-3}+ab^{2n-2}-b^{2n-1}$.

Thus, if $2n = 4$,

$$(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3.$$

§ 98. *f.*) Divide $a^{2n+1} + b^{2n+1}$ by $a + b$.

$$\begin{array}{l|l} a^{2n+1} + b^{2n+1} & a+b \\ \hline a^{2n+1} + a^{2n}b & a^{2n} - \&c. \\ \hline \quad -a^{2n}b + b^{2n+1} = (-a^{2n} + b^{2n})b = (b^{2n} - a^{2n})b. & \end{array}$$

But $b^{2n} - a^{2n}$ is divisible by $a + b$ (§ 97).

$\therefore$ $a^{2n+1} + b^{2n+1}$ is divisible by $a + b$.

Moreover $2n+1$ is an *odd* number, being greater by unity than $2n$, an even number. Hence,

Cor. II. *The* SUM *of any two* ODD *positive integral powers of the same degree is divisible by the* SUM *of their roots.*

$$\frac{a^{2n+1}+b^{2n+1}}{a+b} = a^{2n} - a^{2n-1}b + a^{2n-2}b^2 - a^{2n-3}b^3 + \ldots + a^4b^{2n-4} - a^3b^{2n-3} + a^2b^{2n-2} - ab^{2n-1} + b^{2n}.$$

Thus, if $2n+1 = 5$,

$$(a^5 + b^5) \div (a + b) = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

§ 99. The principles of § 94–98 obviously enable us to resolve another large class (§ 93) of polynomials into their factors. Thus,

$$a^3+b^3 = (a+b)(a^2+ab+b^2).$$

What are the factors of a^4-b^4? of a^5-b^5? of a^5+b^5? of $x^3-27(=x^3-3^3)$? of a^6-x^6? of x^3+64?

THE GREATEST COMMON DIVISOR.

§ 100. A factor common to two or more quantities is called a common divisor; and the greatest common factor, i. e. the product of all the common factors, is called the *greatest common divisor*, or *measure*.

Thus, of the quantities $18abx$ and $20a^2by$, 2, a and b are common divisors; and the greatest common divisor is evidently $2ab$, the product of all the common factors.

NOTES. (1.) The term *divisor* is used here with reference to *perfect* divisibility (§ 82. *g*). (2.) A single factor, as a, which has no integral divisor but itself and unity, is called, as in Arithmetic, a *prime* factor. (3.) Quantities which have no common divisor but unity are called *incommensurable*, or *prime to each other*.

a.) One of two or more quantities may be either multiplied or divided by any factor not found in all the other quantities, without affecting the greatest common divisor. For the factor so introduced or taken out, not being *common* to all the quantities, can form no part of their greatest common divisor.

Thus, the greatest common divisor of $12ax$ and $20ay$ is the same as that of $12ax$ and $20ay\times5b$ or $20ay\div5y$.

§ 101. The greatest common divisor of several monomials must evidently consist of all the common literal factors, multiplied by the greatest common divisor of the numerical coefficients.

What is the greatest common divisor of $12a^3x^4$ and $2a^2x^5$? of $A^2x''y''$ and $B^2x''y''$? of ax and $a'x$? of A^2cy'' and $c^2x''y''$? of $nx^{n-1}(x+1)^n$ and $nx^n(x+1)^{n-1}$.

§ 102. The process of finding the greatest common divisor of two polynomials is substantially the same as that employed in Arithmetic, and depends on the following principle; viz.

The greatest common divisor of two quantities is the same as the greatest common divisor of either of them, and of the remainder obtained by dividing one by the other.

To prove this, let the two quantities be A and B, and divide A by B. Let the integral quotient resulting from this division be Q, and the remainder R. Then $A-BQ=R$, or $A=BQ+R$.

Now every divisor of B is, of course, a divisor of BQ. Therefore every common divisor of A and B is a common divisor of A and BQ. Also, every such common divisor is

a divisor of $A-BQ$ [§ 81][c], i. e. of R. That is, every common divisor of A and B is a common divisor of B and R.

Again every common divisor of B and R will divide BQ and R, and, of course, $BQ+R$ or A. That is, every common divisor of A and B is a common divisor of B and R.

Hence, the greatest common divisor of B and R is the greatest common divisor of A and B.

§ 103. By the same reasoning, if we proceed to divide B by R, and obtain a remainder R', the greatest common divisor of R and R' is the greatest common divisor of B and R, and, therefore, of A and B.

Thus, the greatest common divisor of any of these divisors and its remainder is the greatest common divisor of all the preceding remainders, and also of the original quantities. If then we find a remainder, which divides the preceding remainder, it is the greatest common divisor required.

a.) If the first term of any dividend be not divisible by the first term of the corresponding divisor, we must (1.) suppress any factor of the divisor, not found in the dividend; and (2.) we may, if necessary, multiply the dividend by any factor not found in the divisor (§ 100. *a*).

NOTE. If we suppress in the divisor a factor found also in the dividend, that factor, originally common, will *cease* to be so, and the common divisor will *be less than it ought to be*. If, on the other hand, we introduce into the dividend a factor already found in the divisor, that factor, not originally common, will become so, and the common divisor will be greater than it ought to be (§ 100).

Hence, to find the greatest common divisor of two quantities, we have the following

RULE.

§ 104. *Divide one quantity by the other; then divide the first divisor by the first remainder, the second*

(*c*) If, for instance $A \div D$ and $BQ \div D$ are both whole numbers, their difference or their sum must be a whole number.

divisor by the second remainder, and so on; always rendering the first term of the dividend divisible by the first term of the divisor (§ 103. a). *The divisor which gives no remainder, is the greatest common divisor required.*

a.) If the first remainder which divides the preceding remainder be unity, the quantities are said to have no common divisor, but to be *incommensurable*, or *prime to each* other.

b.) If the greatest common divisor of more than two quantities be required, we must first find that of two of them, and then of that divisor and a third, and so on.

1. Find the greatest common divisor of 98 and 112.

$$\begin{array}{r|l} 112 & 98 \\ 98 & 1 \\ \hline 98 & 14 \\ 98 & 7 \\ \hline \end{array}$$

$\therefore$ 14 is the greatest common divisor required.

NOTE. In Arithmetic, it is, of course, proper to divide the greater number by the less. In Algebra, the quantity containing the highest power of the letter of arrangement will be the first dividend. If the highest power is the same in both, either may be made the dividend.

2. Find the greatest common divisor of x^2+5x+6 and x^2+2x-3.

$$\begin{array}{r|l} x^2+5x+6 & x^2+2x-3 \\ x^2+2x-3 & 1 \\ \hline 3x+9 & \end{array} = 3(x+3). \text{ Reject } 3 \,(103.\ a).$$

$$\therefore \quad \begin{array}{r|l} x^2+2x-3 & x+3 \\ x^2+3x & x-1 \\ \hline -x-3 & \\ -x-3 & \\ \hline \end{array}$$

$\therefore$ $x+3$ is the greatest common divisor.

3. Find the greatest common divisor of $a^2+2ax+x^2$ and a^3-ax^2. *Ans.* $a+x$.

4. Find the greatest common divisor of $9x^3+53x^2-9x-18$ and $x^2+11x+30$. *Ans.* $x+6$.

5. Find the greatest common divisor of $2x^3+x^2-8x+5$ and $7x^2-12x+5$. *Ans.* $x-1$.

6. Find the greatest common divisor of $a^3x+2a^2x^2+2ax^3+x^4$ and $5a^5+10a^4x+5a^3x^2$.

§ 105. *c*.) The application of the above rule (§ 104) to polynomials is simplified in various ways. Thus, before applying the rule,

1. Any factor obviously *common*, may be taken out, and reserved, as a factor of the common divisor required (§ 100).

2. Any factor, found in a part only of the polynomials, may be rejected (§ 100. *a*).

3. If one of two polynomials contain a *letter* not found in the other, the common divisor, obviously, cannot contain that letter, i. e. must be *independent* of it, and must therefore be the common divisor of the coefficients of the several powers of that letter. In this case it is often best to arrange the polynomials with reference to that letter, and to find the greatest common divisor of its coefficients.

NOTE. 1 and 2, above, are most easily applied to monomial factors of the polynomials; for such factors can always be found by inspection (§§ 69, 81). But they are equally applicable to polynomial factors, when we can discover them (§§ 93, 99).

1. Find the greatest common divisor of $a^3+2a^2x+ax^2$ and $5ab^3-5abx^2$. *Ans.* $a(a+x)$.

2. Find the greatest common divisor of $x^3+ax^2+bx^2-2a^2x+abx-2a^2b$ and $x^2+2ax-bx-2ab$.

COMMON MULTIPLE.

§ 106. A COMMON MULTIPLE (§ 46. *a*) of two or more quantities is a quantity which each of them will divide (§ 80. *d*). The *least* common multiple is the *least* quantity which they will divide.

§ 107. A quantity is evidently a multiple of any other quantity of which it contains all the factors; if it contain the factors of each of several quantities, it is their *common* multiple; if it contain each factor no oftener, than some one of the quantities, it is the *least* common multiple. That is,

The least common multiple of several quantities consists of all their factors, each with the highest exponent which it has in any of the quantities.

Thus, the least common multiple of $6ab^2(=2.3ab^2)$ and $9a^2c(=3^2a^2c)$ is $2.3^2a^2b^2c=18a^2b^2c$.

§ 108. The least common multiple of two quantities consists of *all* their prime factors, each with its *greatest* exponent (§ 107); and the greatest common divisor consists of the *common* prime factors, each with its *least* exponent (§ 100). Therefore,

The least common multiple of two quantities is equal to their product divided by their greatest common divisor.

Thus, the greatest common divisor of x^2y and xy^2 is xy; their product is x^3y^3; and their least common multiple is $x^2y^2=x^3y^3\div xy$.

NOTE. Every algebraic factor of the first degree, whether monomial or polynomial, is a prime factor.

PROBLEMS.

§ 109. 1. Given $4a+x=\frac{x^2}{4a+x}$, to find x.

2. Given $\frac{x}{a+b}+\frac{x}{a-b}=2$, to find x.

3. Given $\frac{1}{a^2-x^2}-a=\frac{ax}{a-x}+\frac{a}{a+x}$, to find x.

Ans. $x=\frac{a^2+a^3-1}{a-a^2}$.

4. A sum of money, x, is divided among several persons, so that A receiving \$1000 less than half, and B,

\$1000 more than one third, of the whole, find their portions equal. What is the value of x?

5. Let A receive a less than half, and B, a more than one third, and let their portions be equal. What is the value of x?

6. Two couriers are traveling on the same route, and in the same direction. A is 100 miles in advance of B, and travels 10 miles an hour, while B follows at the rate of 12 miles an hour. In how many hours will they be together?

Let $\quad x =$ the number of hours.

Then $10x =$ the distance A will travel,

and $\quad 12x =$ the distance B will travel, before they come together.

Now, if B overtake A, he must travel as many miles as A, and the distance between them, 100 miles, in addition.

$$\therefore \qquad 12x = 10x + 100; \text{ or } 12x - 10x = 100.$$

$$\therefore \qquad x = 50 \text{ hours.}$$

7. In what time will they be together, if A goes 10 miles an hour, and B 11?

8. In what time, if A goes 10 miles an hour, and B $10\frac{1}{2}$? A 10, and B $10\frac{1}{4}$? A 10, and B $10\frac{1}{10}$? A 10, and B $10\frac{1}{100}$? A 10, and B $10\frac{1}{1000}$? A 10, and B $10\frac{1}{1000000}$?

9. In what time, if A goes 10 miles an hour, and B 10?

In the last case, we have $10x - 10x = (10 - 10)x = 100$.

$$\therefore \qquad x = \frac{100}{10-10} = \frac{100}{0}.$$

a.) How shall this result be interpreted? If we divide 100 by .01, .001, .0001, .00001, &c., the quotient obviously increases as the divisor diminishes, and in the same proportion. Consequently, if the divisor becomes numerically less than any quantity whatever, or 0, the quotient must become greater than any quantity whatever, i. e. *infinite.* For no number can be assigned or conceived, so great as, when multiplied by 0, to produce 100. Hence $\frac{100}{0}$, or, in general, $\frac{a}{0}$ (a being any quantity whatever, numerically

greater than 0), is *infinite*, i. e. *greater than any assignable quantity;* and is expressed by the symbol ∞.

Now, as the difference of the rates, in the preceding problems, became less (i. e. as B gained less in an hour), the number of hours required for him to overtake A became greater. When the difference of the rates is nothing, the time will be infinite (i. e. B will never overtake A). In other words, if B gains nothing in one hour, no number of hours can enable him to gain 100 miles.

10. Again, suppose that A travels 10, and B 8 miles an hour, when will they be together?

Here we have $8x-10x=-2x=100$. $\therefore x=-50$ (§ 5). That is, A and B *were* together 50 hours ago.

NOTE. Had it been proposed to find when they *had been* together, the answer would have been positive (§ 4. *c*).

11. Let A be a miles in advance of B; and let A travel n, and B m miles an hour. When will they be together? *Ans.* In $\frac{a}{m-n}$ hours.

b.) The last problem is the generalization of the preceding (6–10). We shall evidently have, if $a>0$ (§ 6. *a*), when $m>n$, $m-n$ positive, and, of course, the result *positive;* when $m=n$, $m-n=0$, and the result, *infinite;* and when $m<n$, $m-n$, negative, and the result *negative* (§ 10. *d*). If $a=0$, and $m>$, or $<n$, the result is 0 (i. e. they are together now); and if $m=n$, the result is $\frac{0}{0}$ (i. e. they are together now, and must always remain together [§ 109. *c*]).

12. Let them travel towards each other, A, n, and B, m miles an hour. When will they meet?

Ans. In $\frac{a}{m+n}$ hours.

Let $a=100$, $m=12$, and $n=8$; &c.

NOTE. The formula of 12, above, includes this case also. For the rate or velocity of one, being positive, and represented by m, that of the other must be negative (§ 5), and may be denoted by $-n$; and the difference of the rates will be properly expressed by $m-(-n)=m+n$. Hence, we have [12],

$$x=\frac{a}{m-(-n)}=\frac{a}{m+n}.$$

14. The age of a father is 36 years; that of his son is 12. In how many years will the age of the father be just double that of the son. *Ans.* 12.

15. In how many years will it be triple?

Ans. 0 (i. e. it is triple now).

16. In how years will it be quadruple? *Ans.* —4.

That is, it *was* quadruple 4 years ago. If we had inquired, how long *since it had been* quadruple, the result would have been positive.

17. In how many years will the ages be equal?

Ans. ∞ (i. e. they will *never* be equal [§ 109. *a*]).

18. Let A's age be a, and B's, b years; in how many years will A's age be n times B's? *Ans.* $\frac{a-nb}{n-1}$.

Here, if $n>1$, the result will be positive (i. e. the event will be future), when $a>nb$; negative (i. e. the event will be past), when $a<nb$; and zero (i. e. the event will be present), when $a=nb$. If $n=1$, the result will be $\pm\infty$, when $a>$ or $<b$; and the result will be $\frac{0}{0}$, when $a=b$. If $n<1$, the result will be positive, when $a<nb$; negative, when $a>nb$; and zero, when $a=nb$.

c.) In regard to the result $\frac{0}{0}$, it is obvious, that any finite quantity whatever, multiplied by the divisor, 0, will produce the dividend, 0, and is therefore a proper value of the expression. This expression may therefore represent *any quantity whatever,* and is hence called an *indeterminate* expression.

Thus, in problem 12, if $a=0$, and $m=n$ (in which case the result becomes $\frac{0}{0}$), A and B are together now, and must always remain together. Hence, any number of hours whatever will truly express the time at the end of which they will be together. We may, of course, have an *infinite number* of solutions; and, as no one of these is better than another, the problem is said to be *indeterminate.*

CHAPTER III.

FRACTIONS.

§ 110. A FRACTION, in Algebra as in Arithmetic, is the expression of a *division* (§§ 2. *f.* N., 10. *c*).

Thus $\frac{3}{4}$ and $\frac{a}{b}$ express the division of 3 by 4 and of a by b.

§ 111. Again a fraction may be regarded as expressing *equal parts of a unit;* the DENOMINATOR[b] showing the nature of the parts, and the NUMERATOR[c], the number of them employed.

Thus $\frac{3}{4}$ shows, that the unit is divided into 4 equal parts, and that 3 of them are taken. So $\frac{a}{b}$ shows, that the unit is divided into b equal parts, and that a of them are taken.

NOTE. The numerator and denominator are called the *terms* of a fraction.

§ 112. A quantity expressed without the aid of fractions is called *entire* or *integral.* An expression partly entire and partly fractional is called *mixed.*

§ 113. Operations upon fractions are of the same nature in Algebra as in Arithmetic; and depend on the following principles, which we shall here assume without demonstration.

1. *If the numerator of a fraction be multiplied or divided, the fraction itself is equally multiplied or divided.*

2. *If the denominator of a fraction be multiplied or di-*

(*b*) Lat., from denomino, *to name*, because it names the parts.
(*c*) Lat., from numero, *to number*.

vided, the fraction itself is equally divided or multiplied. And, hence,

3. *If the numerator and denominator be either* BOTH *multiplied or* BOTH *divided by the same number, the value of the fraction will not be affected.*

REDUCTION.

§ 114. Let it be required to reduce x to a fraction, whose denominator shall be a.

We have, evidently, $x=\frac{x}{1}=\frac{ax}{a}$. § 113. 3.

Otherwise, $1=\frac{a}{a}$; $\therefore x=\frac{ax}{a}$. §§ 42. *c*, 113. 1.

Hence, to reduce AN ENTIRE QUANTITY TO A FRACTION having a given denominator, we have the following

RULE.

Multiply the quantity by the given denominator, and place the product over the denominator.

Thus, to reduce 3 to fourths, we have $3=\frac{3\times4}{4}=\frac{12}{4}$.

1. Reduce R^2 to a fraction whose denominator is $2bc$.

2. Reduce R^2 to a fraction whose denominator is $\sin b \sin c$.

a.) If there be, connected with the entire quantity, a fractional quantity having the given denominator, we may, obviously, reduce the entire quantity as above, and connect with it by the proper sign the numerator of the given fraction. Thus $4\frac{1}{5}=\frac{20}{5}+\frac{1}{5}=\frac{21}{5}$.

1. Reduce $x-\frac{a^2-x^2}{x}$ to a fractional form.

$$Ans.\ \frac{x^2-(a^2-x^2)}{x}=\frac{x^2-a^2+x^2}{x}=\frac{2x^2-a^2}{x}.$$

2. Reduce $x-\frac{x^2-a^2}{x}$ to a fractional form.

3. Reduce $x+y+\frac{x^2+y^2}{x-y}$ to a fractional form; also $\frac{m}{n}\pm1$ and $\frac{p}{q}\pm1$; also $\frac{A^2}{x}-x$.

§ 115. Reduce $\frac{ab}{a}$ to an entire form.

Divide both numerator and denominator by a (§ 113. 3);

then $$\frac{ab}{a}=b.$$

Hence, to reduce A FRACTION TO AN ENTIRE OR MIXED quantity, we have the following

RULE.

Divide the numerator by the denominator.

NOTE. If the division is exact (§§ 80. d, 82. g), the fraction is reduced to an entire quantity; if not, the fraction can be expressed in an entire form by means of negative exponents (§ 14); or, if the numerator be a polynomial, the fraction will be reduced to a mixed quantity.

1. Reduce $\frac{a^2-x^2}{a-x}$ to an entire quantity. *Ans.* $a+x$.

2. Reduce also $\frac{a^4-x^4}{a+x}$; $\frac{1-\cos^2 v}{1\pm\cos v}$; $\frac{a^2-x^2}{a^2}$.

§ 116. To reduce a fraction TO LOWER TERMS.

RULE.

Divide both numerator and denominator by a common divisor. § 113. 3.

NOTE. To reduce to the *lowest* terms, we must, of course, divide by the *greatest* common divisor (§ 100).

1. Reduce $\frac{1.2.3.4a^3x^5}{3.4.5.6a^4x^4}$ to its lowest terms. *Ans.* $\frac{1.2x}{5.6a}$.

2. Reduce $\frac{a^4-b^4}{a^3-b^3}$ (§ 96); $\frac{a^2-b^2}{a^2\pm2ab+b^2}$ (§ 93).

4. Reduce $\frac{A^2B^2-B^2cx''}{A^2cy''-c^2x''y''}$ to its lowest terms.

Ans. $\frac{B^2}{cy''}$.

5. Reduce $\frac{nx^{n-1}(1+x)^n-nx^n(1+x)^{n-1}}{(1+x)^{2n}}$ to its lowest terms, and simplest form. *Ans.* $\frac{nx^{n-1}}{(1+x)^{n+1}}$.

6. Reduce $\frac{x^3-1}{x^2-1}$ to its lowest terms.

7. Given $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$, to find x.

Ans. $x=\frac{ac}{b}$.

§ 117. To reduce fractions to a COMMON DENOMINATOR.

The value of the fraction must remain unchanged. Consequently, in effecting this reduction, we must either multiply the terms by a common multiplier, or divide them by a common divisor (§ 113. 3). If then the given fractions be already in their lowest terms, the common denominator must be a multiple of each of the given denominators. Hence, the following

RULE.

Multiply all the denominators together for a new denominator, and each numerator by all the denominators except its own, for a new numerator.

a.) Otherwise, *Multiply both terms of each fraction by the denominators of all the other fractions.*

1. Reduce $\frac{2}{3}, \frac{1}{4}$ and $\frac{5}{7}$ to a common denominator.

$$\frac{2\times4\times7}{3\times4\times7}=\frac{56}{84};\quad \frac{1\times3\times7}{4\times3\times7}=\frac{21}{84};\quad \frac{5\times3\times4}{7\times3\times4}=\frac{60}{84}.$$

2. Reduce $\frac{a}{b}, \frac{x}{c}$ and $\frac{x}{b}$ to a common denominator.

$$\frac{a\times c\times b}{b\times c\times b}=\frac{abc}{b^2c};\ \frac{x\times b\times b}{c\times b\times b}=\frac{b^2x}{b^2c};\ \frac{x\times b\times c}{b\times b\times c}=\frac{bcx}{b^2c}.$$

3. Reduce to a common denominator $\frac{u}{v^2}$ and $\frac{1}{v}$.

4. Reduce $\frac{x}{a+b}$ and $\frac{x}{a-b}$ to a common denominator.

5. Reduce $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$, and $\frac{x}{6}$ to a common denominator.

b.) It is evident, that the results obtained by the rule may often be reduced to lower terms, and still have a common denominator. This will be obviated by taking, for a common denominator, the *least* common multiple of the denominators (§ 106). In that case, we must divide the least common multiple by each of the given denominators, and multiply the corresponding numerator by the quotient.

6. Reduce $\frac{x}{a-b}, \frac{x}{a^2-b^2}$ and $\frac{x}{a+b}$ to fractions with the least common denominator.

ADDITION AND SUBTRACTION.

RULE.

§ 118. *Reduce to a common denominator; and then add, or subtract the numerators.*

NOTE. The resulting fractions in the following examples should be reduced to their lowest terms.

$$\frac{a}{b}\pm\frac{c}{b}=\frac{a\pm c}{b}.$$

1. Add $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and $\frac{1}{r}$.

2. $\frac{a}{x}+\frac{a+b}{2(a-x)}-\frac{a+b}{2(a+x)}=$ what?

Ans. $\frac{a^3+bx^2}{x(a-x)(a+x)}=\frac{a^3+bx^2}{a^2x-x^3}$.

3. $\frac{7}{2(x-4)}-\frac{1}{2(x-2)}=$ what? *Ans.* $\frac{3x-5}{x^2-6x+8}$.

5. $\dfrac{a}{2(x-a)^2}+\dfrac{3}{4(x-a)}+\dfrac{1}{4(x+a)}=$ what?

6. $-\dfrac{B^2x''}{A^2y''}-\dfrac{-y''}{c''-x''}=$ what?

MULTIPLICATION AND DIVISION.

§ 119. To MULTIPLY A FRACTION;

RULE.

Multiply the numerator, or divide the denominator, by the multiplier. § 113. 1, 2.

Thus, $\dfrac{ax}{c^2}\times c=\dfrac{acx}{c^2}=\dfrac{ax}{c}$; or, $=axc^{-2}c=axc^{-1}=\dfrac{ax}{c}$ (§ 14).

So $\dfrac{a}{b}\times\dfrac{x}{c}=\dfrac{a\times\frac{x}{c}}{b}=\dfrac{\left(\frac{ax}{c}\right)}{b}=\dfrac{ax}{bc}$; or, $=ab^{-1}xc^{-1}=\dfrac{ax}{bc}$.

a.) The application of the rule to the last example gives the common rule for *multiplying fractions together;*

Multiply the numerators together for the numerator of the product, and the denominators for its denominator.

NOTE. To multiply by $\frac{x}{c}$ is to multiply by x and divide by c. But the fraction is multiplied by multiplying the numerator, and divided by dividing the denominator.

1. Multiply $\dfrac{a+x}{x}$ by $\dfrac{a-x}{x}$.

2. $\dfrac{a+x}{x}\times\dfrac{x}{a+x}=$ what? $-\dfrac{B^2x''}{A^2y''}\times\dfrac{-y''}{c-x''}$?

3. $\dfrac{a^2+ab+b^2}{a^3-a^2b+ab^2-b^3}\times\dfrac{a-b}{a+b}=$ what?

§ 120. To DIVIDE A FRACTION;

RULE.

Divide the numerator or multiply the denominator, by the divisor. § 113. 1, 2.

$$\frac{a}{b}\div c=\frac{\left(\frac{a}{c}\right)}{b}=\frac{a}{bc};\text{ or }\frac{a}{b}\div c=\frac{ab^{-1}}{c}=\frac{a}{bc}\ (\S\ 14).$$

Also, $\frac{a}{b}\div\frac{x}{c}=\frac{a}{b\times\frac{x}{c}}=\frac{ac}{bx}$ (§ 113. 3); or,

$$\frac{a}{b}\div\frac{x}{c}=ab^{-1}\div xc^{-1}=ab^{-1}x^{-1}c=\frac{ac}{bx}\ (\S\ 19.\ \text{Cor. IV.}).$$

a.) Hence, the common rule for *dividing by a fraction; Invert the divisor, and multiply.*

NOTE. To divide any quantity by x divided by c, is the same as to divide c times that quantity by x.

b.) The last rule is otherwise demonstrated thus;

$$\frac{a}{b}\div\frac{x}{c}=\frac{a}{b}\div\left(\frac{1}{c}\times x\right)=\left(\frac{a}{b}\div\frac{1}{c}\right)\div x.$$

But $\frac{a}{b}\div 1=\frac{a}{b}$. $\therefore\ \frac{a}{b}\div\frac{1}{c}=\frac{ac}{b}$ [§ 18. N.].

$$\therefore\qquad \frac{a}{b}\div\frac{x}{c}=\left(\frac{a}{b}\div\frac{1}{c}\right)\div x=\frac{ac}{b}\div x=\frac{ac}{bx}.$$

Apply the same reasoning to $\frac{3}{4}\div\frac{5}{7}$.

c.) Dividing either term of a fraction has the same effect as multiplying the other term. Hence, to divide one fraction by another, we may *divide the terms of the dividend by the corresponding terms of the divisor* (i. e. numerator by numerator, and denominator by denominator).

NOTE. This course is convenient, when the divisions can be exactly performed (§§ 80. *d*, 82. *g*); and it amounts simply to inverting the divisor and canceling equal factors.

1. Divide $\frac{a^4-x^4}{a^3-x^3}$ by $\frac{a^2+x^2}{a-x}$. *Quot.* $\frac{a^2-x^2}{a^2+ax+x^2}$.

2. " $\frac{a^2-2ax+x^2}{a^2-x^2}$ by $\frac{a-x}{a+x}$. *Quot.* 1.

3. $\frac{9}{12}\div\frac{3}{4}=$ what? $\frac{10}{15}\div\frac{2}{5}$? $\frac{1.2.3.4.5x^5}{10.9.8a^6}\div\frac{1.3.5x^2}{9a^3}$?

4. $\left(\frac{a}{a-b}+\frac{b}{a+b}\right)\div\left(\frac{a}{a-b}-\frac{b}{a+b}\right)=$ what?

Ans. $\frac{a^2+2ab-b^2}{a^2+b^2}$.

NOTE. In such examples as the last, it is generally most convenient to reduce the terms of the dividend to a common denominator, and also those of the divisor; and then apply the rule (§ 120. *a*).

d.) $1 \div \frac{x}{y} = \frac{y}{x}$; or $1 \div xy^{-1} = x^{-1}y$ (§ 18) $= \frac{y}{x}$. Hence,

The reciprocal of a fraction is the fraction inverted.

1. $-1 \div -\frac{B^2x''}{A^2y''} =$ what? $-1 \div \frac{p}{y}$? $1 \div \frac{B^2}{A^2}$? $1 \div \frac{3}{4}$?

§ 121. Add k to each term of the fraction $\frac{a}{b}$. Which is the greater, $\frac{a}{b}$ or $\frac{a+k}{b+k}$?

Reducing to a common denominator, we have

$$\frac{a}{b} = \frac{ab+ak}{b^2+bk}; \text{ and } \frac{a+k}{b+k} = \frac{ab+bk}{b^2+bk}.$$

Now the greater fraction has the greater numerator. But the numerators having the term ab common, their relative magnitude depends upon ak and bk, i. e. upon a and b.

$\therefore$ If $\frac{a}{b} < 1$, then $a < b$, $ab+ak < ab+bk$, and $\frac{a}{b} < \frac{a+k}{b+k}$.

If $\frac{a}{b} > 1$, then $a > b$, $ab+ak > ab+bk$, and $\frac{a}{b} > \frac{a+k}{b+k}$.

If $\frac{a}{b} = 1$, then $a = b$, $ab + ak = ab + bk$, and $\frac{a}{b} = \frac{a+k}{b+k}$.

Hence, k being any positive quantity,

$\frac{a}{b} <, >,$ or $= \frac{a+k}{b+k}$, according as $\frac{a}{b} <, >,$ or $= 1$; i. e. as $a <, >,$ or $= b$.

That is, *If any positive quantity be added to both the terms of a fraction, the primitive fraction will be less, greater than, or equal to the new fraction, according as it is less, greater than, or equal to unity.*

Add 100 to the terms of the fractions $\frac{2}{5}$, $\frac{3}{3}$, and $\frac{5}{2}$.

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

TWO UNKNOWN QUANTITIES.

§ **122.** I.) Let $x+y=10$, x and y being both unknown. Here the only condition (§ 38) is, that the *sum* of the unknown quantities shall be 10. Hence we may have $x=0$, $y=10$; $x=1$, $y=9$; $x=7$, $y=3$, &c.; or $x=-1$, $y=11$; $x=-2$, $y=12$, &c.; or $y=-1$, 0, &c., $x=11$, 10, &c.; or $x=\frac{1}{2}$, $y=9\frac{1}{2}$; $x=-\frac{2}{3}$, $y=10\frac{2}{3}$, &c.

II.. Again, let $x-y=4$.

Here the only condition is, that the *difference* of the numbers shall be 4. Hence we may have $x=4$, $y=0$; $x=5$, $y=1$; $x=7$, $y=3$, &c.; or $x=0$, $y=-4$, &c.; or $x=20$, $y=16$, &c.

a.) Either of these equations is, by itself, obviously indeterminate (§ 109. *c*); and may be satisfied (§ 39) by any one of an infinite number of values of x, with corresponding values of y.

b.) But the conditions may be united. That is, it may be required, (1.) that the *sum* of two numbers shall be 10, and (2.) that their *difference* shall be 4. We shall then have, at the same time, $x+y=10$, and $x-y=4$. And the same values of x and y must satisfy both equations.

c.) Now, if $x=9$, we have, by the first condition, $y=1$; and, by the second condition, $y=5$. Thus, the same value of x satisfies both conditions, but the values of y are different. Again, if $y=6$, we have, by the first condition, $x=4$; and, by the second, $x=10$. Here, the same value

of y satisfies both conditions, but the values of x are different. The values in both these cases are said to be *incompatible.*

d.) But the *same* values of BOTH x AND y must satisfy both conditions (*b*). And, in fact, among the values found above (I., II.) there is one set common to the two equations, viz. $x=7$, and $y=3$; thus $7+3=10$, and $7-3=4$.

e.) The solution of the problem consists in finding these *common*, or *compatible* values of x and y.

§ 123. The union (§ 122. *b*) of the two conditions is algebraically expressed by the combination of the equations, treating x and y as symbols of the same quantities in each.

NOTE. It is obvious, that, if x represents the same quantity in the two equations, the sum of x in the first and x in the second, will be $2x$, and their product, x^2. But if x in the second equation denoted a different quantity from x in the first, it might be distinguished as x', and the sum of the two quantities would be $x+x'$, and their product, xx'. The same remark, clearly, applies to y.

ELIMINATION.

BY ADDITION AND SUBTRACTION.

§ 124. Combining (§ 123) the equations

$$x+y=10$$
$$x-y=4$$

by adding them, member by member (Geom. § 22), we have

$$2x=14; \quad \therefore x=7.$$

Substituting, in the first equation ($x+y=10$), for x its value, we have

$$7+y=10; \quad \therefore y=3.$$

These values of x and y introduced into the second equation, $x-y=4$, satisfy it; thus

$7-3=4$, an absolute equation (§ 37. *d*).

NOTES. (1.) The value of y might with equal propriety have been obtained by the substitution of x in the second equation. Thus

$7-y=4$. $\therefore y=3$. (2.) If we had *subtracted* the second equation from the first, we should have found the value of y; and then, by introducing it in either of the equations, we should find x.

§ 125. The solution of the problem in § 124, it will be observed, is effected by *removing* one of the unknown quantities, till the value of the other has been found. This is called ELIMINATION[d]; and the method employed above is called elimination by *addition* and *subtraction*.

1. Given $x+y=15$, ... (1)

$3x+4y=54$. . . . . . . (2)

Multiplying (1) by 3, $3x+3y=45$.

Subtracting the last from (2), $y=9$.

Then from (1), $x+9=15$; $\therefore x=6$.

2. Given $3x-\frac{1}{2}y=-27$, . . (1)

$4x+\frac{1}{3}y=24$. . . (2)

Multiplying (1) by 2, and (2) by 3, we have

$6x-y=-54$, . . (3)

$12x+y=72$. . . (4)

Then, by adding (3) and (4),

$18x=18$. $\therefore x=1, y=60$.

Eliminate x, by dividing (1) by 3 and (2) by 4, and subtracting the first quotient from the second.

§ 126. (1.) *When one of the unknown quantities has the same coefficient in both equations, it can be eliminated, if the signs of the equal coefficients are* ALIKE, *by* SUBTRACTION; *and if* UNLIKE, *by* ADDITION. See §§ 57. 18; 60. 14.

(2.) We may *cause* one of the unknown quantities to have the same coefficient in both equations, by suitably *multiplying* or *dividing* one or both of the equations.

1. Given $5x+6y=40$,

$3x+2y=20$, to find x and y.

Ans. $x=5, y=2\frac{1}{2}$.

(*d*) Lat. elimino, *to turn out of doors.*

2. Given $7x+10y=72$,
$9x+\ 3y=63$, to find x and y.

3. Given $2x-3y=7$,
$x+2y=14$, to find x and y.

BY COMPARISON.

§ 127. 1. Resuming the equations,

$$x+y=10, \quad . \quad . \quad . \quad (1)$$
$$x-y=4, \quad . \quad . \quad . \quad (2)$$

we have, from (1), $x=10-y$,
and from (2), $x=\ 4+y$.

Equating[e] these values of x, we have

$$10-y=4+y. \quad \therefore y=3, \text{ and } x=7.$$

2. Given $2x-3y=7$, and $x+2y=14$.

From the first, $3y=2x-7$; $\therefore y=\frac{2}{3}x-\frac{7}{3}$.
From the second, $2y=14-x$; $\therefore y=7-\frac{1}{2}x$.
$\therefore \quad \frac{2}{3}x-\frac{7}{3}=7-\frac{1}{2}x. \quad \therefore x=8, y=3.$

In this method, *we find from each equation the value of one of the unknown quantities, in terms of the other unknown, and of the known quantities. We equate these two values, and from this new equation find the value of the other unknown quantity; and substitute as before.*

NOTE. This is called elimination by *comparison*, because we *compare* the two values of the unknown quantity.

BY SUBSTITUTION.

§ 128. 1. Taking again the equations

$$x+y=10, \quad . \quad . \quad . \quad (1)$$
$$x-y=4, \quad . \quad . \quad . \quad (2)$$

we have, from (1), $y=10-x$.

(*e*) Lat. æquo, *to make equal*. Quantities are said to be *equated*, or *made equal*, when they are made to constitute the members of an equation.

Substituting in (2), $x-(10-x)=4$. $\therefore x=7, y=3$.

2. Given $$2x-3y=7, \quad . \quad . \qquad (1)$$ $$x+2y=14. \quad . \quad . \qquad (2)$$

From (2), we have $y=7-\frac{1}{2}x$.

Substituting in (1), $2x-3(7-\frac{1}{2}x)=7$;

or $2x-21+\frac{3}{2}x=7$.

$\therefore$ $x=8, y=3$, as above (§ 127. 2).

We here *find, as in the last method, the value of one of the unknown quantities from one equation, and substitute it in the other equation.*

NOTES. (1.) This is called elimination by *substitution*. (2.) Either of the above methods may be employed at pleasure. Sometimes one will be found most convenient, and sometimes another. Practice will enable one to fix upon the best method in each case. It will be useful for the learner, at first, to solve each example by all the methods.

§ 129. 1. Given $\frac{1}{2}x+\frac{1}{4}y=11$,
$\frac{1}{3}x+\frac{1}{5}y=8$, to find x and y.

2. Given $3x-4y=-13$,
$7x-5y=0$, to find x and y.
Ans. $x=5, y=7$.

3. Given $-8x+4y=8$,
$3x+5y=3\frac{1}{2}$, to find x and y.
Ans. $x=-\frac{1}{2}, y=1$.

4. Given $y=2x-4$,
$y=-3x+8$, to find x and y.
Ans. $x=2\frac{2}{5}, y=\frac{4}{5}$.

5. Given $y=ax+b$,
$y=a'x+b'$, to find x and y.
Ans. $x=\dfrac{b-b'}{a'-a}, \quad y=\dfrac{a'b-ab'}{a'-a}$.

With what values of a, a', b and b' will x and y, in this result, become zero? negative? infinite (§ 109. a)? indeterminate (§ 109. c)?

6. Given $ax+by=c$,

$a'x+b'y=c'$, to find x and y.

Ans. $x=\dfrac{b'c-bc'}{ab'-a'b}$, $y=\dfrac{ac'-a'c}{ab'-a'b}$.

7. Given $x+y=S$, and $x-y=D$, to find x and y.

Ans. $x=\frac{1}{2}(S+D)$, $y=\frac{1}{2}(S-D)$. See § 65. 3.

8. A horse and saddle are worth $175; the horse is worth six times as much as the saddle. What is the value of each?

Solve the problem by means of one, and of two unknown quantities.

9. Let the horse and saddle be worth a dollars; and let the horse be worth m times as much as the saddle.

Ans. $\dfrac{a}{1+m}$, value of the saddle; $\dfrac{ma}{1+m}$, that of the horse.

10. A bill of $165 was paid in dollars and eagles, the whole number of pieces being 70. How many were there of each?

Let $x=$ the number of dollars,

$y=$ " eagles.

Then $x+y=70$, and $x+10y=165$.

Or, if $x=$ the number of dollars, then $70-x=$ the number of eagles; &c. Or, let $x=$ the number of eagles; &c.

11. a coins of one kind make a dollar, and b of another kind. How many of each kind must be taken, in order that c pieces may make a dollar?

Let $x=$ the number of the first.

Then either y or $c-x$ will be the number of the second.

Then $\dfrac{x}{a}+\dfrac{y}{b}=1$; or $\dfrac{x}{a}+\dfrac{c-x}{b}=1$.

Ans. $\dfrac{a(c-b)}{a-b}$ of the first kind; $\dfrac{b(a-c)}{a-b}$ of the second.

Let $a=20$, $b=10$; and $c=12, 13, 20, 10, 21, 9$.

Let $a=10$, $b=6$; and $c=8, 9, 10, 6, 5, 11$.

a.) The nature of the question requires whole numbers for the answers. Such values, therefore, should be assigned to a, b, and c, that the numerical values of the above results may be integral. Which of the values above comply with this condition?

b.) With what values of a, b and c, will the above results, or either of them, be positive? negative? zero? infinite? indeterminate? How shall these several results be interpreted?

12. Find a fraction, such that if 1 be added to its numerator, the value will be $\frac{1}{3}$; and if 1 be added to its denominator, the value will be $\frac{1}{4}$.

Let x be the numerator, and y the denominator.

13. A certain number is expressed by two digits whose sum is 9; and if it be increased by five-thirds of itself, the order of the digits will be inverted. What is the number?

Let $\qquad x=$ the left hand digit,

and $\qquad y=$ the right hand digit.

Then $\qquad 10x+y=$ the number, &c.

14. A places a sum of money at interest; B invests \$1000 more than A, at 1 per cent higher interest, and finds his income \$80 more than A's. C invests \$1500 more than A, at 2 per cent higher interest, and receives an income greater than A's by \$150. What are the three sums invested, and at what rates?

Let $x=$ A's sum, and $y=$ his rate of interest per cent.

Then $\qquad \dfrac{xy}{100}=$ his income, &c.

MORE THAN TWO UNKNOWN QUANTITIES.

§ 130. I. Let $x+y+z=10$ . . . . . (1),
x, y and z being all unknown.

Here we may assign any value we please to any one of the unknown quantities, and still have an infinite number of values for the other two; or we may assign any values

whatever to two of them, and find a corresponding value for the third. Thus, the problem is *doubly* indeterminate.

II. Again, let $2x-y+3z=7$ . . . (2)

This equation is equally indeterminate as the first. And if we unite the two conditions, we may still assign any value we please to one of the unknown quantities, and deduce corresponding common values for the other two. Or, eliminating one of the unknown quantities, we shall have a single equation with two unknown quantities.

Thus, adding (1) and (2),

$$3x+4z=17 \quad . \quad . \quad . \quad (a)$$

Hence, the problem is still indeterminate (§ 122. a).

III. But again, let

$$3x+2y+4z=27 \quad . \quad . \quad . \quad (3)$$

Multiplying (1) by 2,

$$2x+2y+2z=20 \quad . \quad . \quad . \quad (b)$$

Subtracting (b) from (3),

$$x+2z=7 \quad . \quad . \quad . \quad (c)$$

Combining (a) and (c), [§ 123], we find $x=3$, $y=5$, and, then substituting in (1), (2) or (3), $z=2$.

Note. We might, obviously, have employed either of the other methods of elimination (§ 124–128).

§ 131. Hence, to find the common values of three unknown quantities, from three equations, *we eliminate one of the unknown quantities from all the equations, thus forming two equations with two unknown quantities.* We then solve these equations by § 124–128.

1. Given $\frac{1}{2}x+\frac{1}{3}y+\frac{1}{6}z=12$,
$x-y+z=12$,
$2x+3y-4z=12$, to find x, y and z.

2. Given $x+\frac{1}{2}y+\frac{1}{3}z=27$
$x+\frac{1}{3}y+\frac{1}{4}z=20$
$x+\frac{1}{4}y+\frac{1}{5}z=16$, to find x, y, and z.

Ans. $x=1$, $y=12$, $z=60$.

§ 132. We have found *one* equation with *two* unknown quantities, and *one* or *two* equations with *three* unknown quantities to be indeterminate. In like manner, if we had *three* equations with *four* unknown quantities, by eliminating one of the unknown quantities, we should have *two* equations containing *three* unknown quantities, and, of course, indeterminate. By like reasoning, we shall find, that *any number whatever of equations must be indeterminate, if the number of unknown quantities is greater than the number of* INDEPENDENT *equations.*

NOTES. (1). *Independent* equations are those, *of which no one is implied by the rest.* Thus $x+y=3$, and $3x+3y=9$ are *not* independent, because one is a necessary consequence of the other. (2.) When a number of equations containing several unknown quantities are spoken of, they must be understood to be independent equations, unless the contrary is stated, or clearly implied by the connection.

§ 133. If we have *four* equations involving *four* unknown quantities, the elimination of one of the unknown quantities will result in *three* equations containing *three* unknown quantities, which may be solved by § 131. The same reasoning will obviously extend to *any* number of equations containing an *equal* number of unknown quantities.

§ 134. Hence, to find the value of any number of unknown quantities from an equal number of equations, *we eliminate one of the unknown quantities from all the equations, thus diminishing by one, at the same time, the number of equations and of unknown quantities contained in them.*

We then eliminate, from the new equations, another unknown quantity, and so on, till we arrive at a single equation containing one unknown quantity.

1. Given $7x-2z+3u=17$,

$$4y-2z+t=11,$$
$$5y-3x-2u=8,$$

$$4y-3u+2t=9,$$
$$3z+8u=33,$$ to find x, y, z, u and t.

Ans. $x=2$, $y=4$, $z=3$, $u=3$, $t=1$.

2. Given $2x-3y+2z=13$,
$$2u-x=15,$$
$$2y+z=7,$$
$$5y+3u=32,$$ to find x, y, u and z.

Ans. $x=3$, $y=1$, $u=9$, $z=5$.

3. The sum of four numbers is 25. Half of the first number is equal to twice the second, and to three times the third; and the fourth is four times the third. What are the numbers?

Let u, x, y and z represent the numbers.

Also let x represent one of the numbers, and solve the problem with one unknown quantity.

4. Find three numbers, such, that the sum of the first and second shall be 15; the sum of the first and third, 16; and the sum of the second and third, 17.

Solve the above problem by one, by two, and by three unknown quantities.

5. A, B and C form a partnership. A contributes a certain sum; B contributes a times, and C, b times as much as A; and the whole stock is c. How much did each contribute? See § 55. 4.

Ans. $\frac{c}{1+a+b}$, A's part; $\frac{ac}{1+a+b}$, B's part, &c.

5. A and B can perform a piece of work in 8 days; A and C, in 9 days; and B and C, in 10 days; in how many days could each person, alone, perform the same work?

Let x, be the number of days required by A; y, by B; and z, by C.

Then, in 1 day, A will perform $\frac{1}{x}$ of the work; B, $\frac{1}{y}$; and C, $\frac{1}{z}$. But A and B together perform, in 1 day, $\frac{1}{8}$ of the work; &c.

$$\therefore \qquad \frac{1}{x}+\frac{1}{y}=\frac{1}{8}, \quad \frac{1}{x}+\frac{1}{z}=\frac{1}{9}, \quad \text{and } \frac{1}{y}+\frac{1}{z}=\frac{1}{10}.$$

NOTE. Instead of clearing of fractions, regard $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$ as the unknown quantities; and from their values, when found, find the values of x, y and z (§ 50).

Ans. A in $14\frac{34}{49}$ days; B, in $17\frac{23}{41}$; and C, in $13\frac{7}{31}$.

6. Let A and B perform the work in a days; A, and C, in b days; and B and C, in c days; and find the general expression for the time in which each person, alone, would perform the work.

Ans. $\dfrac{2abc}{bc+ac-ab}$, A's time; $\dfrac{2abc}{bc+ab-ac}$, B's; and $\dfrac{2abc}{ab+ac-bc}$, C's.

§ 135. We have seen, that, when the number of unknown quantities is *greater* than the number of *independent* equations, the problem is indeterminate. When, on the other hand, the number of unknown quantities is *less* than the number of *independent* equations, the equations are *inconsistent* in their conditions, and cannot all be satisfied by the same values of the unknown quantities. For, if the values found from two equations containing two unknown quantities would satisfy a third, this would be implied by the rest, and, of course, would not be independent of them (§ 132. N. 1). E. g. the equations $x+y=10$, $x-y=4$, and $2x+y=40$, are obviously inconsistent.

§ 136. When a single equation containing more than one unknown quantity is considered by itself, the unknown quantities are frequently called *variables;* and one of them is said to be a *function* (§ 26) of the rest.

a.) Thus, in the equations $2y+3x=10$, $y=ax+b$, y is a function of x, and x of y; or, as it is usually expressed, $y=F(x)$, and $x=F(y)$. For, if we give any value whatever to one of these quantities, we can deduce a corresponding value for the other; and, if we vary the value of the first, the value of the second undergoes a corresponding change.

b.) If an equation contain more than two unknown quantities, each of them is a function of all the rest. Thus, in the equation $2x+3y+z=75$, we have $x=F(y, z)$ [i. e. x a function of y and z], $y=F(x, z)$, and $z=F(x, y)$.

NOTES. (1.) Of two quantities, that of which the other is said to be a function, is called the *independent variable*. (2.) Either of the unknown quantities may, obviously, be made the independent variable, and the other will be the function. (3.) If there are more than two variables, one may be regarded as a function of all the rest, they being all independent; or one may be a function of the second, the second, of the third, and so on, the last only being independent.

§ 137. Arithmetic, in its ordinary applications, furnishes only positive and definite solutions. It is, therefore, sometimes said, that negative, infinite and indeterminate results do not furnish a proper answer to a question. The answers which they furnish would indeed not be intelligible to one unacquainted with the algebraic language. But to one familiar with that language a negative result answers a question as directly and intelligibly as a positive; an infinite, as a finite. See §§ 4. 9, 10; 109. 10, 16.

Thus, when we inquire, how long it will be before a certain event will take place, we equally answer the question by saying that it will take place in 12 years (§ 109. 14), or in no years (i. e. now, § 109. 15), or that it took place 4 years ago (§ 109. 16), or that it will never take place (§ 109. 9, 17), or that it is taking place all the time (§ 109. *c*).

In like manner, if we inquire how far east a certain point lies, we equally answer the question by saying an infinite distance, a finite distance (as 10 miles), no distance, a distance west, or that such a point exists every where in a line running east and west.

Arithmetic does not ordinarily take cognizance of infinite or indeterminate results; and, regarding numbers simply as such, without respect to their character as positive or negative (§ 8), its questions must be proposed in such a manner, that an answer may be expressed by a number simply.

§ 138. It will be observed, that between the positive and negative values, we always have a value equal to zero or to infinity, i e. equal to 0 or $\frac{a}{0}$. See §§ 4. 6–10; 109. 8–10, *b*, 14–16. That is, between the positive and negative results, there is one, either equal to 0, or whose denominator has become 0.

§ 139. We have $\frac{a}{0}=\infty$ (§ 109. *a*). That is,

(1.) *A finite quantity divided by zero is equal to infinity.*

Also (§ 42. *c*), $\qquad a=0\times\infty$. That is,

(2.) *Zero multiplied by infinity is equal to a finite quantity.*

Again (§ 42. *d*), $\qquad \dfrac{a}{\infty}=0$. That is,

(3.) *A finite, divided by an infinite quantity is equal to zero.*

NOTE. We arrive at the idea of infinity by continually diminishing a divisor, and thus finding a greater and greater quotient (§ 109. *a*). Hence 0 is sometimes said to denote an *infinitely small* quantity, or an *infinitesimal* (i. e. a quantity *less than any assignable quantity* [§ 109. *a*]).

§ 140. The expression $\frac{0}{0}$ is *not always* indeterminate. For, instead of the whole numerator and denominator, a *common factor* may have been reduced to 0. If then this common factor be removed (§ 113. 3), the expression will no longer be indeterminate. Thus, when $b=a$,

$$\frac{a^2-b^2}{a-b}=\frac{0}{0}. \quad \text{But } \frac{a^2-b^2}{a-b}=\frac{(a+b)(a-b)}{a-b}=a+b=2a,$$

when $b=a$.

So, if $b=a$, $\dfrac{a^n-b^n}{a-b}=\dfrac{0}{0}$. But, performing the division, and then making $b=a$, we have $\dfrac{a^n-b^n}{a-b}=na^{n-1}$ (§ 96. *b*).

1. $\dfrac{x^3-1}{x-1}=$ what, when $x=1$? *Ans.* 3.

2. $\dfrac{(x-y)^2}{x^2-y^2}=$ what, if $x=y$? *Ans.* $\dfrac{0}{x+y}=0$.

3. $\frac{x^2-y^2}{(x-y)^2}=$ what, if $x=y$? *Ans.* $\frac{x+y}{0}=\infty$.

CHAPTER V.

INEQUALITIES.

§ 141. Two quantities, connected by the sign $<$ or $>$ (§ 2. *b*), constitute an INEQUALITY. An inequality may be called *increasing*, or *decreasing*, according as the second member is greater or less than the first. When two inequalities *both increase*, or *both decrease*, they may be said to *have the same tendency*, or to subsist in the same sense or direction; otherwise, they are of *contrary* tendency.

§ 142. Operations upon *inequalities* are similar to those upon equations, and depend chiefly upon an analogous *axiom* (§ 42); viz.

UNEQUAL *quantities, equally affected, remain* UNEQUAL.

Hence, if *equal* quantities be (1.) *added to*, or (2.) *subtracted from*, both sides of an *inequality*, or if both sides be (3.) *multiplied*, or (4.) *divided by equal* quantities, *the results will be unequal.*

§ 143. In transforming an inequality, however, we must not only *preserve the inequality*, but we must, at every step, determine *which way it tends* (i. e. which member is the greater). Hence, the necessity of observing the following obvious principles.

§ 144. *a.*) *If equal quantities be* ADDED TO, *or* SUBTRACTED FROM, *both members of an inequality, the tendency of the inequality will always remain unchanged.* Thus,

$$10>6;\ 10\pm8>6\pm8$$

$$-10<-6;\ -10\pm12<-6\pm12.\quad \text{See § 6. } a.$$

NOTE. Hence, *transposition* applies to *inequalities*, in like manner as to equations. Thus,

$$10-5>12-8.\ \therefore\ 10>17-8.$$

So, if $y^2+x^2-R^2>0$, then $y^2+x^2>R^2$, and $y^2>R^2-x^2$.

§ 145. *b.*) With still greater reason,

If two inequalities, having the same tendency, be added, member by member, there will result an inequality of the same tendency.

Thus, $9>7$ and $-1>-3$. $\therefore 8>4$.

So if $a>b$, and $m>n$, then $a+m>b+n$.

NOTE. If one inequality be *subtracted*, member by member, from another of the same tendency, the result will *not always be an inequality*; nor, *if it be, will it necessarily have the same tendency.*

§ 146. *c.*) *If the members of an inequality be subtracted from the same number, the tendency of the inequality will be changed.* Thus,

$$8>6, \text{ and } 10-8<10-6.$$

In like manner, $0-8<0-6$ (i. e. $-8<-6$). Hence,

d.) *If the signs of both members be changed, the tendency will be changed.*

NOTE. This results directly from the principle, that, of negative quantities, that which is numerically the greatest is absolutely the least (i. e. leaves the least remainder). See § 6. *a.*

§ 147. *e.*) *If both members of an inequality be* MULTIPLIED *or* DIVIDED *by the same* POSITIVE *number, the resulting inequality will have the same tendency; if by the same* NEGATIVE *number, the tendency will be changed.* Thus,

$$6>-8;\ \text{and } 6\times3>-8\times3, \text{ or } 18>-24.$$

But $6\times-3<-8\times-3$, or $-18<+24$.

So $6\div2>-8\div2$, or $3>-4$.

But $6\div-2<-8\div-2$, or $-3<+4$.

Also, if $a>b$, $ak>bk$ (§ 121), but $-ak<-bk$.

§ 148. *f.*) Hence, an inequality may *always be cleared of fractions.* For, if we multiply by a positive denominator, the tendency remains the same; if by a negative, it is changed. Or, if the denominator is negative, we may place its sign before the fraction, and then multiply by the positive denominator (§§ 68. *b*, 80. *b*).

§ 149. *g.*) *If the members of an inequality be* POSITIVE, *and be both raised to the same* POSITIVE INTEGRAL *power of any degree whatever, the tendency of the inequality will remain unchanged.*

Thus, $7>3$; $7^2>3^2$;

NOTE. This holds equally of *fractional* powers or *roots* (§ 23. *b*), so long as we confine ourselves to their positive values (23. *f.* 1). If we regard the negative values of an even root, the tendency is, of course changed.

§ 150. *h.*) *Whatever be their signs, if the members of an inequality be both raised to the same* ODD *positive power, the tendency will remain unchanged.* Thus,

$$-3<2;\ (-3)^3<2^3.$$

CHAPTER VI.

POWERS AND ROOTS.

MONOMIALS.

§ 151. To raise a MONOMIAL TO ANY POWER;

Multiply the exponent of each factor by the exponent of the required power. (§ 24. *d*).

a.) This rule depends on the obvious principle, that *a*

power of a product is equal to the product of the same powers of the several factors. Thus,

$$(abc)^n = a^n b^n c^n; \ (abc)^2 = a^2 b^2 c^2; \ (ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}}.$$

b.) This rule applies equally to numerical and literal factors; and, so far as Algebra is concerned, it is sufficient. It is proper, however, to perform upon the numerical coefficient the arithmetical operations indicated by its exponent. Thus, if its exponent be positive and integral, raise the coefficient to the arithmetical power denoted by the exponent; if the exponent be positive and fractional, raise the coefficient to the power denoted by the numerator and extract the root denoted by the denominator; if the exponent be negative, perform the same operations as if it were positive, and place the result in the denominator of a fraction of which the other factors of the monomial constitute the numerator.

c.) The sign of an *even* integral (§ 22. *d*) power is *positive;* the sign of an *odd* integral power is the same as that of its base (§ 22. N.). See § 11. N. 2.

1. What is the fourth power of $2ab^2x^{\frac{1}{2}}y^{-\frac{1}{3}}$?

Ans. $2^4a^4b^8x^{\frac{4}{2}}y^{-\frac{4}{3}} = 16a^4b^8x^2y^{-\frac{4}{3}}$.

2. $(-3a^2b^{-1})^3 =$ what? $(-a^2b^2cx)^2$? $(a^nx^{n-3})^2$?

3. $(na^{-2}x^{-\frac{1}{2}})^{\frac{1}{2}} =$ what? $(a^{-2}x^{-\frac{1}{2}})^{-\frac{5}{2}}$? $(10^{\frac{3}{10}})^5$? $(a^nx^r)^{\frac{1}{p}}$? $(u^{\frac{r}{s}})^{s-1}$?

4. $(a^{\frac{m}{n}}x^{\frac{p}{q}})^{\frac{r}{s}} =$ what? $(3a^{-2}b^{\frac{1}{2}}x^2y^{-\frac{1}{2}})^{-5}$? $(R^2x^{-2})^{-\frac{5}{2}}$?

d.) In determining the sign of a *fractional* (§ 22. *d*) power, its exponent should be reduced to its lowest terms. Then, if the *numerator* of the exponent is an *even* number, the power is *positive;* if the *denominator* is *even,* the power of a *positive* quantity is *ambiguous* (i. e. $\pm$), and of a *negative* quantity, *imaginary* (§ 23. *f*); and if *both numerator and denominator* are *odd* numbers, the power has the same sign as the quantity itself.

e.) A power of a *fraction* is found by raising both numerator and denominator to the required power (§§ 119. *a*, 120. *c*). Or, all the factors of the denominator may be carried into the numerator, if we change at the same time the signs of their exponents (§§ 14, 17); and then the quantity may be treated like any other monomial.

1. $\left(\frac{x}{a}\right)^{\frac{3}{4}}=$ what? $\left(\frac{2a^2x}{3bc^2}\right)^3$? $\left(\frac{x^2}{R^2}\right)^{\frac{5}{2}}$? $\left(\frac{x^m}{a^m}\right)^{-p}$? $\left(\frac{B^2}{A}\right)^2$?

ROOTS.

§ 152. From the preceding rule (§ 151), we deduce the following specific rule, in which the term root is used in the same sense as in Arithmetic.

To extract any *root of a monomial;*

Extract the root of the numerical coefficient as in Arithmetic; and divide the exponent of each literal factor by the number of the root.

a.) This rule is obviously included in the preceding (§ 25). But for convenience, and on account of the very frequent necessity of extracting the square and cube roots, it is given here in a distinct form.

b.) An *odd* root of a *positive* quantity is *positive;* of a *negative* quantity, *negative* (§ 23. *e*).

c.) *An even* root of a *positive* quantity is *either positive or negative* (§ 23. *f.* 1).

d.) An *even* root of a *negative* quantity is *imaginary* (22. *f.* 2).

1. What is the square root of $25a^2bc^{-1}x^{\frac{2}{3}}$?

Ans. $\sqrt{(25a^2bc^{-1}x^{\frac{2}{3}})}=(25a^2bc^{-1}x^{\frac{2}{3}})^{\frac{1}{2}}=5ab^{\frac{1}{2}}c^{-\frac{1}{2}}x^{\frac{1}{3}}$.

2. $\sqrt{(49a^3b^2x^{-3})}=$ what? $(100a^{-4}b^mx^{2n})^{\frac{1}{2}}$? $(x^2y)^{\frac{1}{4}}$?

3. $\sqrt[3]{-\frac{27a^6x^9}{64b^4y^{12}}}=$ what? $\left(\frac{8a^2x^3}{27b^3z}\right)^{\frac{1}{3}}$? $\left(\frac{\sqrt{ab}}{\sqrt[4]{c\sqrt{r}}}\right)^{\frac{1}{2}}$?

§ 153. *Any* root of *any* monomial can be algebraically expressed, but it is not always possible to perform exactly the arithmetical operations upon the coefficient. Thus the square root of $2ab^2$ is $2^{\frac{1}{2}}a^{\frac{1}{2}}b$; but the exact arithmetical computation of $2^{\frac{1}{2}}$ cannot be attained. Such a root is called *incommensurable*[f], *irrational*[g] or *surd*[h]. A numerical quantity whose root can be exactly found is called a *perfect power*.

NOTE. It will be shown hereafter, that, if a root of a whole number is not a whole number, it cannot be expressed at all except by approximation.

§ 154. The use of the term *perfect power*, as applied to algebraic monomials, is sometimes restricted to the cases in which *the numerical coefficient is a perfect power, and each exponent is divisible* (§ 80. *d*) *by the number of the root.* The roots of all quantities which are not perfect powers are called *irrational*, *radical* (§ 23. *d.* N.) or *surd* quantities.

§ 155. A radical quantity can frequently be *reduced to a simpler form.* Thus,

$$(192a^3b^2c)^{\frac{1}{2}} = (64a^2b^2 \times 3ac)^{\frac{1}{2}} = 8ab(3ac)^{\frac{1}{2}}.$$

$$(108a^5b^6x)^{\frac{1}{3}} = (27a^3b^6 \times 4a^2x)^{\frac{1}{3}} = 3ab^2(4a^2x)^{\frac{1}{3}}.$$

We here separate the root into two factors, one of which is rational (i. e. expressed by integral exponents), while the other is radical (i. e. expressed by fractional exponents). This can, obviously, be done, whenever, after the extraction of the root, any of the exponents are improper fractions; or, when, before the extraction, any of them are greater than the number of the root, and not exact multiples of it.

§ 156. We shall, evidently, effect this simplification, if,

(*f*) Lat. in, *not*, con, *together* and mensura, *measure;* having *no common measure* (§ 100) with unity. (*g*) Lat. in, *not* and ratio, *relation*, *ratio;* whose ratio to unity cannot be exactly expressed. (*h*) Lat. surdus, *that is not heard;* because it cannot be expressed.

in extracting the root, we divide the exponent of each letter by the number of the root, and *set the integral part of the quotient as an exponent of the letter in one factor, and the fractional part as an exponent of the same letter in another factor.* If the root has been extracted, we have only to *reduce all the improper fractions* among the exponents to *mixed numbers, and set each letter under its integral exponent in one factor, and under its fractional exponent in another.*

1. Reduce $\sqrt{60a^3b^4x}$ to its simplest form.

Ans. $(60a^3b^4x)^{\frac{1}{2}} = (4.15)^{\frac{1}{2}}aa^{\frac{1}{2}}b^2x^{\frac{1}{2}} = 2ab^2(15ax)^{\frac{1}{2}} = 2ab^2\sqrt{15ax}.$

2. Reduce $(75a^2b^5x^7)^{\frac{1}{2}}$ to its simplest form.

3. Reduce also $\sqrt[3]{54a^8x^3}$; $\sqrt{32x^2y^5}$; $\sqrt[4]{a^8bpc^{-2}x}$; $\sqrt{(2p)}x^{\frac{3}{2}}$?

4. Separate $a^{\frac{5}{2}}b^{\frac{7}{3}}c^{\frac{5}{4}}x^{\frac{m+n}{n}}$ into rational and radical factors.

Ans. $a^2b^2cx \times a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}x^{\frac{m}{n}} = a^2b^2cx \times a^{\frac{6n}{12n}}b^{\frac{4n}{12n}}c^{\frac{3n}{12n}}x^{\frac{12m}{12n}} = a^2b^2cx\sqrt[12n]{a^{6n}b^{4n}c^{3n}x^{12m}} = a^2b^2cx(a^{6n}b^{4n}c^{3n}x^{12m})^{\frac{1}{12n}}.$ See § 160.

§ 157. *a.*) In simplifying an irrational *fraction*, it is generally best to multiply both numerator and denominator by a multiplier which will make the denominator rational. Thus, we may simplify the fraction $\left(\frac{3}{7}\right)^{\frac{1}{2}}$, as follows:

$$\left(\frac{3}{7}\right)^{\frac{1}{2}} = \frac{3^{\frac{1}{2}}}{7^{\frac{1}{2}}} = \frac{3^{\frac{1}{2}}7^{\frac{1}{2}}}{7^{\frac{1}{2}}7^{\frac{1}{2}}} = \frac{(3.7)^{\frac{1}{2}}}{7} = \frac{1}{7}(21)^{\frac{1}{2}}.$$

So
$$\left(\frac{2}{3}\right)^{\frac{1}{3}} = \frac{2^{\frac{1}{3}}}{3^{\frac{1}{3}}} = \frac{2^{\frac{1}{3}}3^{\frac{2}{3}}}{3^{\frac{1}{3}}3^{\frac{2}{3}}} = \frac{(2.3^2)}{3} = \frac{1}{3}(18)^{\frac{1}{3}}.$$

NOTE. *If the sum of the exponents of each letter in two monomials be an integer, the product will, of course, be rational.*

§ 158. *b.*) Every *negative* quantity can, obviously, be re-

garded as containing the factor, -1, together with a positive factor.

Thus $-a = a(-1)$; $-a^2 = a^2(-1)$; $-25 = 25(-1)$.

Hence, $(-a)^{\frac{1}{2}} = a^{\frac{1}{2}}(-1)^{\frac{1}{2}}$; $(-a^2)^{\frac{1}{2}} = (a^2)^{\frac{1}{2}}(-1)^{\frac{1}{2}} = a\sqrt{-1}$.

1. $(-B^2)^{\frac{1}{2}} =$ what? *Ans.* $B\sqrt{-1}$.

2. $(-25a^2bx^{\frac{1}{3}})^{\frac{1}{2}} =$ what? *Ans.* $5ab^{\frac{1}{2}}x^{\frac{1}{6}}\sqrt{-1}$.

Hence, *every even root of a negative quantity consists of a real quantity multiplied by* $\sqrt{-1}$.

NOTE. Such expressions as the above must not be regarded as having any actual value whatever. One factor is real, but the other is imaginary; and the product is, of course, imaginary.

§ 159. Addition, subtraction, multiplication and division are, of course, performed upon irrational quantities according to the general rules. In addition and subtraction, it is frequently more convenient to separate the quantities into their rational and radical factors, and reduce the resulting polynomials by § 33. *c.*

§ 160. After the separation of the rational and radical factors of a monomial, it is frequently convenient to reduce all the fractional exponents to a common denominator, and, writing only the numerator of each exponent over its letter, enclose the whole in a parenthesis under the reciprocal of the denominator as an exponent; or, if preferred, place the whole under a radical sign with the common denominator over it. See § 156. 4.

NOTES. (1.) Radicals which have the same quantities, both numerical and literal, under the same fractional exponent or radical sign, are called *similar* radicals. (2.) The rational factor, multiplied by a radical, is, of course, properly called the *coefficient* of the radical.

§ 161. The *rational* factors may be placed under the radical exponent or sign, if their exponents be reduced to fractions having the common denominator. This is commonly

called *carrying the coefficient of the radical under the sign.* Thus,

$$x\sqrt{a}=x^{\frac{2}{2}}a^{\frac{1}{2}}=(ax^2)^{\frac{1}{2}}, \text{ or } \sqrt{(ax^2)}.$$

1. $x(2Rx)^{\frac{1}{2}}=$ what? *Ans.* $(2Rx^3)^{\frac{1}{2}}$, or $\sqrt{(2Rx^3)}$.

2. $x(2Rx)^{\frac{1}{3}}=$ what? *Ans.* $(2Rx^4)^{\frac{1}{3}}$, or $\sqrt[3]{(2Rx^4)}$.

This transformation is particularly useful in finding an approximate root of a number. Thus,

$$7\sqrt{5}=7\times 2 \text{ (the nearest unit)}=14. \quad \text{But}$$

$7\sqrt{5}=7^{\frac{2}{2}}.5^{\frac{1}{2}}=(7^2.5)^{\frac{1}{2}}=(49.5)^{\frac{1}{2}}=(245)^{\frac{1}{2}}=16$ (the nearest unit).

NOTE. In extracting the root of 5 and multiplying by 7, we multiply the *error* in the root by 7. In the other process, we avoid this source of inaccuracy.

REMARK. In some, especially of the earlier treatises, the radical *sign* is used almost to the exclusion of fractional exponents. The exponent, however, is much more convenient, and many of the difficulties connected with the *calculus of radicals*, as it is called, disappear, when the *exponent* takes the place of the *sign*. Hence, if it is intended to use the radical sign in expressing the result, it is still generally best to employ the exponent in the operations by which the result is obtained.

IMAGINARY QUANTITIES.

§ 162. The expression $\sqrt{-1}$ may be taken as the representative of all imaginary quantities. The treatment of imaginary quantities will be best illustrated by considering some of the powers of $\sqrt{-1}$. Thus,

$$(\sqrt{-1})^2=(-1)^{\frac{1}{2}}.(-1)^{\frac{1}{2}}=-1.$$

$$(\sqrt{-1})^3=(-1)^{\frac{2}{2}}(-1)^{\frac{1}{2}}=-1(-1)^{\frac{1}{2}}=-\sqrt{-1}.$$

$$(\sqrt{-1})^4=(-1)^{\frac{4}{2}}=(-1)^2=1.$$

$$(\sqrt{-1})^5=\sqrt{-1};\ (\sqrt{-1})^6=-1;\ (\sqrt{-1})^7=-\sqrt{-1};$$
$$(\sqrt{-1})^8=1;\ \text{\&c.}$$

Hence, $(\sqrt{-a^2})^2=(a\sqrt{-1})^2=a^2\times -1=-a^2.$

$$\sqrt{-a^2}\ \sqrt{-b^2}=a\sqrt{-1}\times b\sqrt{-1}=ab\times -1=-ab.$$

$$\sqrt{-a^2}\sqrt{-b^2}\sqrt{-c^2}=-abc\sqrt{-1};\ \&c.$$

NOTES. (1.) Care must be taken not to confound *imaginary* with *irrational* expressions. A numerical *surd*, as $\sqrt{2}$, cannot be exactly expressed in units or parts of a unit, but we may approximate as near as we please to its true value. An *imaginary* expression, on the other hand, as $\sqrt{-1}$, has *no actual value*, and we can, of course, make no approach to its value; nor can one quantity be said to come any nearer to its true value than another. Thus, no quantity can be conceived, which, multiplied into itself, will produce -1; and the expression $\sqrt{-1}$ is merely a symbol of an impossible operation; a symbol, to which there exists no corresponding quantity. (2.) It may be thought, that such symbols, not representing quantity, can be of no utility, and should have no place in investigations relating to quantity. But some of the most remarkable and useful results of algebraic reasoning depend upon the presence of imaginary symbols. (3.) An imaginary result generally indicates, that we have, in some way, introduced inconsistent conditions into our investigation; and demonstrates the impossibility of finding, under the circumstances, such a result as we, at first, proposed to find.

POLYNOMIALS.

§ 163. We shall consider here only the *positive integral* powers, and simple roots of polynomials. It is evident, moreover, that if we can find such powers and roots of a polynomial, we can find all powers. For the formation of the power denoted by the numerator, and the extraction of the root denoted by the denominator will give any *positive fractional* power; and the proper combination of those processes with division will give all *negative* powers.

§ 164. The most obvious method of finding a positive integral power of any quantity is by continued multiplication of the quantity by itself; taking it as a factor as many times as there are units in the exponent of the power. Thus we have already found (§ 89)

$$(a+x)^2=(a+x)(a+x)=a^2+2ax+x^2.$$

So $(a+x)^3=(a+x)(a+x)(a+x)=a^3+3a^2x+3ax^2+x^3.$

$(a+x)^4=(a+x)^3(a+x)=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$

$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$

§ 165. We find that, in these instances, (1.) the first term of each power of the *binomial*, $a+x$, is that power of the first term of the binomial; (2.) that the exponents of the first or *leading* quantity, a, diminish, and those of x increase by unity in the successive terms; (3.) that the exponent of a in the last term is zero, and that of x is the exponent of the required power; (4.) that the numerical coefficient of the second term is the same as the exponent of the required power; and (5.) that the numerical coefficients at equal distances from the two extremities of the series are equal.

NOTE. It will be shown hereafter, that these principles apply to all positive integral powers of a binomial, and that all but the third and fifth apply to every power of a binomial, whether the exponent be positive or negative, integral or fractional.

§ 166. We have enunciated these principles as proved only so far as we have found them true by actual multiplication. Let us suppose, that we have found the law of the first and second terms, given above (§ 165. 1, 4), to be true to the nth power. See § 95. N. 1.

Then we have $(a+x)^n = a^n + na^{n-1}x + \&c.$

Multiplying by $a+x$,

$$(a+x)^{n+1} = a^{n+1} + (n+1)a^n x + \&c.$$

If then the principles 1, and 4 of § 165 are true for the nth power, they are true for the $n+1$ power, and so on, without limit, n being *any positive integer* whatever.

§ 167. If we substitute, in the above expressions (§ 164), $-x$ for $+x$, we shall, evidently, obtain the powers of $a-x$. This substitution will, obviously, cause all the terms containing the odd powers of x to become negative, and will occasion no other change. Thus,

$(a-x)^2 = a^2 - 2ax + x^2$. See § 90.

$(a-x)^3 = a^3 - 3a^2x + 3ax^2 - x^3$.

$(a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$.

§ 168. $(a+x)^2 = a^2 + 2ax + x^2$. Substitute $b+y$ for x.

Then $(a+b+y)^2=a^2+2a(b+y)+(b+y)^2$ . . (1)

or $(a+b+y)^2=(a+b)^2+2(a+b)y+y^2$ . . . (2)

Developing, $(a+b+y)^2=a^2+2ab+b^2+2ay+2by+y^2$.

That is, *The square of the sum of three numbers is equal to the sum of their squares, plus twice the sum of their products, taken two and two.*

NOTE. By increasing the number of terms, we might find similar expressions for the square of any polynomial. Thus,

$$(a+b+c+z)^2=(a+b+c)^2+2(a+b+c)z+z^2.$$

Hence, *The square of any polynomial is equal to the sum of the squares of the terms, plus twice the sum of their products, taken two and two.*

§ 169. $(a+x)^3=a^3+3a^2x+3ax^2+x^3$. Substitute $b+y$ for x. Then

$(a+b+y)^3=a^3+3a^2(b+y)+3a(b+y)^2+(b+y)^3$. (1)

or $(a+b+y)^3=(a+b)^3+3(a+b)^2y+3(a+b)y^2+y^3$. (2)

$\therefore$ $(a+b+y)^3=a^3+3a^2b+3ab^2+b^3+3a^2y+6aby+3b^2y$ $+3ay^2+3by^2+y^3=a^3+b^3+y^3+3a^2(b+y)+3b^2(a+y)$ $+3y^2(a+b)+6aby.$

That is, *The cube of a trinomial is equal to the sum of the cubes of the terms, plus three times the square of each term into the sum of the other two, plus six times the product of the three terms.*

NOTES. (1.) We might find, in like manner, expressions for the *higher* powers of a trinomial. (2.) If one of the terms of the *trinomial* becomes zero, the formulæ of §§ 168, 169 give the square and cube of a *binomial.*

SQUARE ROOT OF A POLYNOMIAL.

§ 170. Find the square root of $a^2+2ab+b^2$.

a.) The polynomial being arranged according to the descending powers of a, we know, that a^2 must be the square of one term of the root (§§ 73. 1; 82. *a*, *b*).

b.) We know, moreover, that the polynomial contains,

besides the square of the first term, twice the product of the first term by the second (§ 168), and so on. If, therefore, we divide the next term of the arranged polynomial by $2a$, we shall find another term of the root.

c.) If now we subtract from the given polynomial the square of the terms of the root already found, the remainder, if there be one, will contain the terms which resulted from the multiplication of the remaining terms of the root by each other, and by the terms already found (§ 168. 2, N.).

d.) We may, therefore, find another term of the root, by dividing the first term of the arranged remainder by twice the first term of the root; and so on (§ 82. *c*).

Thus,

$$\begin{array}{r|l} a^2+2ab+b^2 & \underline{a+b} \\ \underline{a^2\qquad\qquad} & \\ 2ab+b^2 & 2a+b \\ \underline{2ab+b^2} & \end{array}$$

NOTES. (1.) It will be seen, that we have subtracted the square of the two terms of the root found (§ 170. *c*). For, $(a+b)^2 = a^2+2ab+b^2 = a^2+(2a+b)b$. Now we subtracted a^2 at first, and afterwards subtracted $(2a+b)b$. (2.) Also, after each subtraction, we shall have subtracted the square of the whole root then found (§ 171. Ex. 1, *a*).

(3.) As there is no remainder, there can be no other terms in the root. And whenever we find a remainder equal to zero, the work is completed (§ 82. *g*), and the given polynomial may be said to be a *perfect power*.

(4.) If, however, after exhausting the given terms of the polynomial, we still have a remainder, the root cannot be exactly found by this process.

(5.) We may, however, continue the process, and develop the root in an infinite series, as in division (§ 87).

From the reasoning above, we deduce the following

RULE.

§ 171. 1. *Arrange the polynomial according to the powers of some letter.*

2. *Extract the root of the first term for the first*

term of the required root; and subtract its square from the given polynomial.

3. *Double the part of the root already found, for a partial divisor; and divide the first term of the remainder by the first term of the doubled root; setting the quotient, with its proper sign, as a term both of the root and of the divisor.*

4. *Multiply the divisor thus completed by the new term of the root, and subtract the product. Continue the process as long as the case may require.*

1. $(9x^4-12x^3+16x^2-8x+4)^{\frac{1}{2}}=$ what?

$$\begin{array}{l|l}
9x^4-12x^3+16x^2-8x+4 & \underline{3x^2-2x+2} \\
\underline{9x^4} & \\
\quad -12x^3 & \underline{6x^2-2x} \\
\quad \underline{-12x^3+4x^2} & \\
\qquad\qquad 12x^2 & \underline{6x^2-4x+2} \\
\qquad\qquad \underline{12x^2-8x+4} &
\end{array}$$

a.) We must be careful, at each step, to double *the whole* of the root already found, for a divisor. For

$$(a+b+c)^2=(a+b)^2+2(a+b)c+c^2. \qquad \S\, 168.\ 2.$$

Also, $(a+b+c+x)^2=(a+b+c)^2+2(a+b+c)x+x^2$; and so on. § 168. N.

2. What is the square root of $a^4+4a^3b+6a^2b^2+4ab^3+b^4$? *Ans.* $a^2+2ab+b^2$.

3. $(16x^4+24x^3+89x^2+60x+100)^{\frac{1}{2}}=$ what?

4. $(a\pm2a^{\frac{1}{2}}b^{\frac{1}{2}}+b)^{\frac{1}{2}}=$ what?

Ans. $a^{\frac{1}{2}}\pm b^{\frac{1}{2}}$, or $\sqrt{a}\pm\sqrt{b}$.

5. $(a^2+2ax^{\frac{1}{2}}+x)^{\frac{1}{2}}=$ what? $(x^2+px+\frac{1}{4}p^2)^{\frac{1}{2}}$?

6. $(a^{2n}\pm2a^nx^n+x^{2n})^{\frac{1}{2}}=$ what? *Ans.* $a^n\pm x^n$.

§ 172. *b.*) The square root of a trinomial perfect power may be immediately determined by inspection. For the

roots of the terms containing the highest and the lowest powers of the letters being extracted, the remaining term must contain twice the product of those roots (§ 89). Moreover, if this double product of the roots is positive, they must have like signs; if negative, unlike. Hence, to extract the root of a trinomial perfect power, *extract the roots of the terms containing the highest and the lowest powers of the letters, and give them like or unlike signs, according as the remaining term is positive or negative.* See § 93. I. II.

$$(a^2 \pm 2ax + x^2)^{\frac{1}{2}} = a \pm x.$$

$(n^2 \pm 2n + 1)^{\frac{1}{2}} =$ what? $(64a^2 + 112ab + 49b^2)$?

§ 173. *c.*) 1. Extract the square root of a^2+x^2. § 170. N. 4.

$$\begin{array}{l|l}
a^2+x^2 & a+\frac{1}{2}a^{-1}x^2-\frac{1}{8}a^{-3}x^4+\&c. = a\left(1+\frac{1}{2}a^{-2}x^2-\frac{1}{8}a^{-4}x^4+\&c.\right). \\
a^2 & \\
\hline
\quad x^2 & 2a+\frac{1}{2}a^{-1}x^2 \\
\quad x^2+\frac{1}{4}a^{-2}x^4 & \\
\hline
\quad\quad -\frac{1}{4}a^{-2}x^4 & 2a+a^{-1}x^2-\frac{1}{8}a^{-3}x^4 \\
\quad\quad -\frac{1}{4}a^{-2}x^4-\frac{1}{8}a^{-4}x^6+\frac{1}{64}a^{-6}x^8 & \\
\hline
\quad\quad\quad \frac{1}{8}a^{-4}x^6-\frac{1}{64}a^{-6}x^8 &
\end{array}$$

Here the second term of the root (§ 171. 3) is $x^2 \div 2a = \frac{1}{2}a^{-1}x^2 = \frac{x^2}{2a}$ (§ 80. *a*). Thus, in another form,

$$(a^2+x^2)^{\frac{1}{2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \&c. = a\left(1 + \frac{x^2}{2a^2} - \frac{x^4}{8a^4} + \frac{x^6}{16a^6} - \frac{5x^8}{128a^8} + \&c.\right).$$

d.) Otherwise, $(a^2+x^2)^{\frac{1}{2}} = (a^2)^{\frac{1}{2}}(1+a^{-2}x^2)^{\frac{1}{2}} = a(1+a^{-2}x^2)^{\frac{1}{2}}$. See §§ 155, 181.

Substitute y for $a^{-2}x^2$, and extract the root of $1+y$. We thus find $(1+y)^{\frac{1}{2}} = 1+\frac{1}{2}y-\frac{1}{8}y^2+\frac{1}{16}y^3-\frac{5}{128}y^4+\&c.$

Then substituting for y its value, and multiplying by a, we have the same result as before.

Let $a=10$, and $x=1$; then $(a^2+x^2)^{\frac{1}{2}}=(101)^{\frac{1}{2}}=10(1+\frac{1}{200}-\frac{1}{80,000}+\&c.)=10(1+.005-.0000125+\&c.)=10(1.0049875+\&c.)=10.049875$ &c.

2. $(R^2-x^2)^{\frac{1}{2}}=$ what?

$$\textit{Ans. } R(1-\frac{1x^2}{2R^2}-\frac{1x^4}{8R^4}-\frac{1\,x^6}{16R^6}-\&c.)$$

§ 174. In like manner, in Arithmetic, we extract the square root of the greatest square contained in the left hand period; subtract the square; divide the remainder by twice the part of the root found; set the new figure, at the same time, in the root and in the divisor; and multiply the divisor so completed by the new figure of the root.

1. Extract the square root of 55696.

5′56′96	200+30+6	or		5′56′96 | 236
4 00 00				4
1 56 96	400+30		43)	156
1 29 00				129
27 96	460+6		466)	2796
27 96				2796

a.) The *terms* are not distinct in Arithmetic as in Algebra. But it is evident, that the square of the unit figure must be found in the first and second places on the right; the square of the tens, in the third and fourth places; and the square of the hundreds, in the fifth and sixth.

b.) Hence, if we separate the number into such periods of two figures each, the square of the highest figure of the root will be contained in the left hand period; and, when subtracted, will leave twice the product of that figure by the other figures, together with the squares of the other figures.

c.) Moreover, the double product of the first and second figures of the root, together with the square of the second figure will, obviously, be contained in what remains of the first two periods on the left.

d.) Consequently, we must divide the remainder of the first two periods by twice the first figure of the root, regarded as denoting tens; and add to the partial divisor the figure thus obtained for a complete divisor.

e.) Then the remainder of the first two periods, together with the third, will, obviously, contain the double product of the two figures of the root, already found, by the third figure, together with the square of the third; and so on.

NOTES. (1.) The terms of the power not being distinct, the double of the part of the root already found is only a trial divisor; and the correctness of the next figure of the root can be verified only by multiplying it into the complete divisor, and subtracting the product. (2.) The trial divisor, on account of the local value of figures, forms a large part of the complete divisor, and is therefore an approximation to it. (3.) $(a+1)^2-a^2=2a+1$. If, therefore, the remainder is not less than twice the root found, *plus* one, the last figure is too small.

§ 175. *f.*) If after obtaining the last integral figure of the root we have not a remainder equal to zero, the given number is not a perfect square; and *its root cannot be found but by approximation.* For, if a mixed number (§ 112) could express the exact root of a whole number, the mixed number being reduced to an improper fraction whose terms (§ 111. N.) are prime to each other, the square of this fraction must be a whole number.

But, if two numbers are prime to each other, the product of any number of factors equal to the one will, evidently, be prime to the product of any number of factors equal to the other. For such a combination of *prime* factors can never introduce a *common* factor. Consequently any power whatever of the numerator will be prime to the same power of the denominator; and the square of the improper fraction which we supposed to be the root of the whole number, must be an irreducible fraction, and not a whole number. Hence *no irreducible fraction can be the root of a whole number;* and, if the root of a whole number is not a whole number, it *cannot be expressed at all* except by approximation (§ 153. N.).

§ 176. *g.*) The approximation to a surd (§ 153) root is effected on the same principle as the simplification of a radical fraction (§ 157); i. e. by reducing the number to a fraction, whose denominator is a perfect power. Thus,

$$2^{\frac{1}{2}}=\left(\frac{2\times5^2}{5^2}\right)^{\frac{1}{2}}=\left(\frac{50}{25}\right)^{\frac{1}{2}}=\frac{7}{5}, \text{ within } \tfrac{1}{5}.$$

Or, $2^{\frac{1}{2}}=\left(\frac{2\times10^2}{10^2}\right)^{\frac{1}{2}}=\left(\frac{200}{100}\right)^{\frac{1}{2}}=\frac{14}{10}=1.4$, within $\frac{1}{10}$.

Again $2^{\frac{1}{2}}=\left(\frac{2\times100^2}{100^2}\right)^{\frac{1}{2}}=\left(\frac{20,000}{10,000}\right)^{\frac{1}{2}}=1.41$, within .01.

The greater the denominator, the closer, obviously, is the approximation. For, the root of the numerator being extracted to the nearest unit, the root of the fraction is found within a unit divided by the root of the denominator.

h.) The approximation is, of course, most conveniently performed with the powers of 10, 100, 1000, &c. And this is the ordinary process of approximation in Arithmetic, in which the denominator is not written; and the approximation may be carried to any extent, by annexing new periods of cyphers to the number (i. e. by multiplying it repeatedly by 10^2), and thus extending the root to additional places of decimals (i. e. dividing the root repeatedly by 10).

i.) In like manner, if the terms of a vulgar fraction are not perfect powers, we can generally extract its root most conveniently, by first reducing it to a decimal. If it reduces to a repeating decimal, instead of annexing cyphers in approximating we should, of course, annex figures of the repetend.

CUBE ROOT OF A POLYNOMIAL.

§ 177. Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

a.) Reasoning as in respect to the square root (§ 170. *a*), we arrange the polynomial, extract the cube root of the first term, and subtract the cube.

b.) We then know, that the first term of the arranged remainder will consist of three times the square of the first term of the root into another term. We may, therefore, find another term of the root by dividing the first term of the remainder by three times the square of the first term of the root. See § 169.

c.) If now we subtract from the given polynomial the cube of the part of the root already found, the first term of the arranged remainder, if there be one, will contain three times the square of the first term of the root into another term (§ 169); and so on.

d.) The cube of $a+b$ consists, besides a^3 already subtracted, of $3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)b$. The most convenient method, therefore, of completing the subtraction of $(a+b)^3$, is, after having found b by dividing the first term of the remainder by $3a^2$, to form the polynomial factor $3a^2+3ab+b^2$, and then multiply it by b. That is, we may add to three times the square of the first term, three times the product of the two terms, and the square of the new term; and multiply the sum by the new term.

Thus,
$$\begin{array}{r|l} a^3+3a^2b+3ab^2+b^3 & a+b, \textit{ Root.} \\ a^3 & \\ \hline 3a^2b+3ab^2+b^3 & 3a^2+3ab+b^2, \textit{ Divisor.} \\ 3a^2b+3ab^2+b^3 & \\ \hline \end{array}$$

NOTES. (1.) We have, evidently, subtracted the cube of the two terms of the root. For, $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+(3a^2+3ab+b^2)b$. (2.) Remarks similar to § 170. N. 2, 3 4, 5 apply equally to the cube root. But the *approximation* by this process to the cube roots of imperfect powers is so laborious, that other methods, which will be considered hereafter, are preferable.

From the reasoning above we have the following

RULE.

§ 178. 1. *Arrange the polynomial according to the powers of some letter.*

2. *Extract the cube root of the first term, for the first term of the root, and subtract its cube.*

3. *Divide the first term of the arranged remainder by three times the square of the first term of the root.*

4. *Add to three times the square of the part of the root previously found, three times the product of the previous part of the root by the new term, and also the square of the new term* (§ 177. *d*).

5. *Multiply the divisor so completed by the new term of the root; subtract, multiply the square of the whole root already found by* 3, *divide, complete the divisor, multiply and subtract; and continue the process as long as the case may require.*

1. $(t^6-6t^5+15t^4-20t^3+15t^2-6t+1)^{\frac{1}{3}}=$ what?

$t^6-6t^5+15t^4-20t^3+15t^2-6t+1$	t^2-2t+1, *root.*
t^6	
$-6t^5+15t^4-20t^3+15t^2-6t+1$	$3t^4-6t^3+4t^2$, 1*st*
$-6t^5+12t^4-\ 8t^3$	*divisor.*
$3t^4-12t^3+15t^2-6t+1$	$3t^4-12t^3+15t^2-6t$
$3t^4-12t^3+15t^2-6t+1$	$+1$, 2*d divisor.*

2. $(a^6+3a^4x^2+3a^2x^4+x^6)^{\frac{1}{3}}=$ what?

3. $(\frac{1}{27}x^3+\frac{2}{3}x^2+4x+8)^{\frac{1}{3}}=$ what? *Ans.* $\frac{1}{3}x+2$.

4. $(a^{\frac{3}{2}}\pm\frac{3}{2}a+\frac{3}{4}a^{\frac{1}{2}}\pm\frac{1}{8})^{\frac{1}{3}}=$ what? *Ans.* $a^{\frac{1}{2}}\pm\frac{1}{2}$.

§ 179. In like manner, in Arithmetic, the number being separated into periods of three figures each, (because the cube of the unit figure must, evidently, be found in the first three places, the cube of the tens in the next three, and so on,) we extract the root of the greatest perfect cube in the left hand period; subtract the cube from that period; divide the remainder of that period, with the next, by three times the square of the first figure of the root regarded as standing in the place of tens; then complete the divisor, multiply, subtract, bring down the next period; and divide

by three times the square of the whole root already found, regarded as denoting tens, and so on.

What is the cube root of 1953125?

$$
\begin{array}{l|l}
1'953'125 & \underline{125} \\
1 & \\
\hline
\quad 953 & \underline{300+60}+4=364,\ 1st\ complete\ divisor. \\
\quad 728 & \\
\hline
\quad 225125 & \underline{43200+1800+25}=45025,\ 2d\ complete\ divisor. \\
\quad 225125 & \\
\hline
\end{array}
$$

NOTES. (1.) The terms of the power not being distinct, three times the square of the part of the root already found is only a trial or approximate (§ 174. N. 2) divisor, and the correctness of the next figure of the root can be verified only by multiplying it into the completed divisor, and subtracting the product. (2.) $(a+1)^3-a^3=3a^2+3a+1$. If, therefore, the remainder is not less than three times the square of the root found, *plus* three times the root, *plus* one, the last figure is too small.

N^{TH} ROOT OF A POLYNOMIAL.

§ 180. We know that $(a+b)^n=a^n+na^{n-1}b+$&c. (§ 166). Hence we have, for finding the nth root of a polynomial, the following

RULE.

1. *Arrange the polynomial, extract the n^{th} root of the first term for the first term of the root, and subtract its power.*

2. *Divide the first term of the arranged remainder by n times the $(n-1)$ power of the first term of the root.*

3. *Raise the whole root so found to the n^{th} power and subtract it.*

4. *Divide the first term of the arranged remainder by the same divisor as before, subtract the n^{th} power of the whole root from the given polynomial, and so on.*

a.) If we make $n=2$, we have a rule for the square root; if $n=3$, for the cube root.

1. $(a^5-10a^4x+40a^3x^2-80a^2x^3+80ax^4-32x^5)^{\frac{1}{5}}=$ what?

$$\begin{array}{l|l}
a^5-10a^4x+40a^3x^2-80a^2x^3+80ax^4-32x^5 & \underline{a-2x} \\
a^5 & \\
\hline
\quad -10a^4x \;|\; 5a^4, \textit{ divisor.} & \\
\hline
a^5-10a^4x+40a^3x^2-80a^2x^3+80ax^4-32x^5. & \\
\hline
\end{array}$$

2. $(16a^4+96a^3x+216a^2x^2+216ax^3+81x^4)^{\frac{1}{4}}=$ what?

Ans. $2a+3x$.

b.) In the last example, the root may be more easily found by extracting the *square root twice*. And, in general, whenever the number of the root is a product of two or more numbers, we may extract successively the roots indicated by the several numbers.

Thus, to find the sixth root, we may extract the square root, and then the cube root; to find the eighth root, we may extract the square root three times; and so on.

c.) It is best in such cases, if the roots are of different degrees, as the square and cube roots, to extract the lowest root first.

§ 181. There is frequently an advantage in *simplifying* (§ 155) the expression of a root of a binomial, or of any polynomial which is not a perfect power. See § 173. *d.* Thus,

$$(a^3-a^2x)^{\frac{1}{2}}=(a^2)^{\frac{1}{2}}(a-x)^{\frac{1}{2}}=a(a-x)^{\frac{1}{2}}.$$

$$(a^3+2a^2x+ax^2)^{\frac{1}{2}}=(a^2+2ax+x^2)^{\frac{1}{2}}a^{\frac{1}{2}}=(a+x)a^{\frac{1}{2}}.$$

$$(a^5-a^3x^2)^{\frac{1}{3}}=a(a^2-x^2)^{\frac{1}{3}}.$$

SQUARE ROOT OF $a\pm b^{\frac{1}{2}}$.

§ 182. The square root of a binomial of the form $a\pm b^{\frac{1}{2}}$ can sometimes be obtained by a peculiar process, which depends on the following principles.

§ 183. I. Let a and x be rational, and $\sqrt{b}$ and $\sqrt{y}$, irrational; then if $a \pm \sqrt{b} = x \pm \sqrt{y}$, a will be equal to x, and $\sqrt{b}$ to $\sqrt{y}$.

For, if a be not equal to x, let it be equal to $x \pm c$.

Then $\quad x \pm c \pm \sqrt{b} = x \pm \sqrt{y}$; or $c \pm b^{\frac{1}{2}} = y^{\frac{1}{2}}$.

$\therefore$ Squaring $\quad c^2 \pm 2cb^{\frac{1}{2}} + b = y$.

$\therefore \quad b^{\frac{1}{2}} = \pm \dfrac{y - c^2 - b}{2c}$ (§ 42. a, d); that is, an irrational, equal to a rational quantity, which is absurd (§ 175). See Geom. § 23. Hence,

Two binomials, consisting each of a rational and of an irrational term, cannot be equal, unless the rational terms are equal to each other, and also the irrational.

§ 184. Let $(a + b^{\frac{1}{2}})^{\frac{1}{2}} = x^{\frac{1}{2}} + y^{\frac{1}{2}}$, one or both of the quantities $x^{\frac{1}{2}}$ and $y^{\frac{1}{2}}$ being irrational, and x and y monomial.

Then, squaring,

$$a + b^{\frac{1}{2}} = x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y; \text{ or } a + \sqrt{b} = x + 2\sqrt{(xy)} + y.$$

$$\therefore \quad a = x + y, \text{ and } b^{\frac{1}{2}} = 2x^{\frac{1}{2}}y^{\frac{1}{2}} \text{ (§ 183).}$$

Hence, subtracting,

$$a - b^{\frac{1}{2}} = x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = (x^{\frac{1}{2}} - y^{\frac{1}{2}})^2.$$

$\therefore \quad (a - b^{\frac{1}{2}})^{\frac{1}{2}} = x^{\frac{1}{2}} - y^{\frac{1}{2}}$ (§ 52. N.). That is,

If $\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}$, *then* $\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}$.

Thus, $\quad (3 + 5^{\frac{1}{2}})^2 = 9 + 6 \times 5^{\frac{1}{2}} + 5 = 14 + 6 \times 5^{\frac{1}{2}}$;

and $\quad (3 - 5^{\frac{1}{2}})^2 = 9 - 6 \times 5^{\frac{1}{2}} + 5 = 14 - 6 \times 5^{\frac{1}{2}}$.

$\therefore \quad \sqrt{(14 + 6\sqrt{5})} = 3 + \sqrt{5}$, and $\sqrt{(14 - 6\sqrt{5})} = 3 - \sqrt{5}$.

$$(2^{\frac{1}{2}} \pm 3^{\frac{1}{2}})^2 = 2 \pm 2 \times 2^{\frac{1}{2}}.3^{\frac{1}{2}} + 3 = 5 \pm 2 \times 2^{\frac{1}{2}}.3^{\frac{1}{2}}.$$

$$\therefore \quad \sqrt{(5 \pm 2\sqrt{(2 \times 3)})} = \sqrt{2} \pm \sqrt{3}.$$

Note. x and y being monomials, the squares of $\sqrt{x}$ and $\sqrt{y}$ must be rational, and will, of course, combine by addition, into a single rational term a; while their double product, being equal to

$\sqrt{b}$ (§ 181), is irrational, and will be positive or negative according as $\sqrt{x}$ and $\sqrt{y}$ have the same or different signs.

§ 185. Now assume $(a+b^{\frac{1}{2}})^{\frac{1}{2}}=x^{\frac{1}{2}}+y^{\frac{1}{2}}$ (1);

then $(a-b^{\frac{1}{2}})^{\frac{1}{2}}=x^{\frac{1}{2}}-y^{\frac{1}{2}}$ (2). § 184.

Squaring (1) and (2),

$$a+b^{\frac{1}{2}}=x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y, \text{ and}$$

$$a-b^{\frac{1}{2}}=x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y.$$

Adding, and dividing by 2, we have

$$a=x+y \ (3).$$

Again, multiplying together (1) and (2), we have

$$(a^2-b)^{\frac{1}{2}}=x-y \ (4).* \qquad \text{§ 92.}$$

Hence, from (3) and (4),

$$x=\frac{a+(a^2-b)^{\frac{1}{2}}}{2}, \text{ and } y=\frac{a-(a^2-b)^{\frac{1}{2}}}{2}.$$

$$\therefore x^{\frac{1}{2}}=\left(\frac{a+(a^2-b)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \text{ and } y^{\frac{1}{2}}=\left(\frac{a-(a^2-b)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}.$$

Hence, substituting in (1) and (2),

$$(a+b^{\frac{1}{2}})^{\frac{1}{2}}=\left(\frac{a+(a^2-b)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}+\left(\frac{a-(a^2-b)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}.$$

$$(a-b^{\frac{1}{2}})^{\frac{1}{2}}=\left(\frac{a+(a^2-b)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}-\left(\frac{a-(a^2-b)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}.$$

Or, putting $(a^2-b)^{\frac{1}{2}}=c$, we have

$$(a+b^{\frac{1}{2}})^{\frac{1}{2}}=\left(\frac{a+c}{2}\right)^{\frac{1}{2}}+\left(\frac{a-c}{2}\right)^{\frac{1}{2}},$$

or $$\sqrt{(a+\sqrt{b})}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}};$$

* $(a+b^{\frac{1}{2}})^{\frac{1}{2}}(a-b^{\frac{1}{2}})^{\frac{1}{2}}=[(a+b^{\frac{1}{2}})(a-b^{\frac{1}{2}})]^{\frac{1}{2}}$ [§ 151. a] $=(a^2-b)^{\frac{1}{2}}$ [§ 92].

$$(a-b^{\frac{1}{2}})^{\frac{1}{2}}=\left(\frac{a+c}{2}\right)^{\frac{1}{2}}-\left(\frac{a-c}{2}\right)^{\frac{1}{2}},$$

or
$$\sqrt{(a-\sqrt{b})}=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}.$$

NOTE. These expressions will, evidently, not reduce to a convenient form, unless $(a^2-b)^{\frac{1}{2}}$ is rational, i. e. unless a^2-b is a perfect square.

The above results may be verified by squaring. Thus,

$$(a\pm b^{\frac{1}{2}})^{\frac{1}{2}}=\left(\frac{a+c}{2}\right)^{\frac{1}{2}}\pm\left(\frac{a-c}{2}\right)^{\frac{1}{2}}.$$

$$\therefore \quad a\pm b^{\frac{1}{2}}=\left[\left(\frac{a+c}{2}\right)^{\frac{1}{2}}\pm\left(\frac{a-c}{2}\right)^{\frac{1}{2}}\right]^2=\frac{a+c}{2}\pm(a^2-c^2)^{\frac{1}{2}}$$
$$+\frac{a-c}{2}=a\pm(a^2-c^2)^{\frac{1}{2}}=a\pm(a^2-(a^2-b))^{\frac{1}{2}}=a\pm b^{\frac{1}{2}}.$$

1. $(3+2\sqrt{2})^{\frac{1}{2}}=$ what?

Here $a=3$, and $b^{\frac{1}{2}}=2(2)^{\frac{1}{2}}=(2^2.2)^{\frac{1}{2}}=8^{\frac{1}{2}}$ (§ 161).

$$\therefore \quad c=(a^2-b)^{\frac{1}{2}}=(9-8)^{\frac{1}{2}}=1^{\frac{1}{2}}=1.$$

$$\therefore \quad \left(\frac{a+c}{2}\right)^{\frac{1}{2}}+\left(\frac{a-c}{2}\right)^{\frac{1}{2}}=\left(\frac{3+1}{2}\right)^{\frac{1}{2}}+\left(\frac{3-1}{2}\right)^{\frac{1}{2}}=2^{\frac{1}{2}}+1.$$

We may verify this result by squaring $2^{\frac{1}{2}}+1$. Thus,

$$(2^{\frac{1}{2}}+1)^2=2+2(2)^{\frac{1}{2}}+1=3+2\sqrt{2}.$$

2. $(9\pm4\times5^{\frac{1}{2}})^{\frac{1}{2}}=$ what? *Ans.* $2\pm5^{\frac{1}{2}}$.

3. $(7\pm2\times10^{\frac{1}{2}})^{\frac{1}{2}}=$ what? *Ans.* $5^{\frac{1}{2}}\pm2^{\frac{1}{2}}$.

4. $(\frac{5}{6}\pm(\frac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}=$ what? *Ans.* $(\frac{1}{2})^{\frac{1}{2}}\pm(\frac{1}{3})^{\frac{1}{2}}$.

5. $\sqrt{(6+6\sqrt{-3})}=$ what?

Here $a=6$, and $b^{\frac{1}{2}}=6(-3)^{\frac{1}{2}}=(6^2(-3))^{\frac{1}{2}}=(-108)^{\frac{1}{2}}$.

$\therefore b=-108$, and $c=(a^2-b)^{\frac{1}{2}}=(36-(-108))^{\frac{1}{2}}=(144)^{\frac{1}{2}}$
$=12$.

$$\therefore \quad (6+6(-3)^{\frac{1}{2}})^{\frac{1}{2}}=3+(-3)^{\frac{1}{2}}.$$

6. $\sqrt{(2+4\sqrt{-2})}+\sqrt{(2-4\sqrt{-2})}=$ what? *Ans.* 4.

7. $[bc+2b(bc-b^2)^{\frac{1}{2}}]^{\frac{1}{2}}+[bc-2b(bc-b^2)^{\frac{1}{2}}]^{\frac{1}{2}}=\pm2b.$

8. $[ab+4c^2-d^2+2(4abc^2-abd^2)^{\frac{1}{2}}]^{\frac{1}{2}}=$ what?

Ans. $\sqrt{ab}+\sqrt{4c^2-d^2}$.

§ 186. Such expressions as $a\pm\sqrt{b}$, or $\sqrt{a}\pm\sqrt{b}$, are sometimes called *binomial surds.*

We have $(a+b^{\frac{1}{2}})(a-b^{\frac{1}{2}})=a^2-b$ (§ 92).

Also $(a^{\frac{1}{2}}\pm b^{\frac{1}{2}})(a^{\frac{1}{2}}\mp b^{\frac{1}{2}})=a-b$. Hence,

The product of the sum and difference of two square roots, or of a square root and a rational quantity will be rational.

Thus, $(2+\sqrt{5})(2-\sqrt{5})=4-5=-1$.

§ 187. *a*). This principle is frequently useful in freeing one of the terms of a fraction, or one of the members of an equation, of irrational expressions. Thus, let it be required to reduce $\frac{2-\sqrt{3}}{2+\sqrt{3}}$ to a fraction having a rational denominator.

We have $\frac{(2-\sqrt{3})\times(2-\sqrt{3})}{(2+\sqrt{3})\times(2-\sqrt{3})}=\frac{(2-\sqrt{3})^2}{4-3}=7-4\sqrt{3}$.

1. Reduce in like manner $\frac{5}{\sqrt{8}\pm\sqrt{3}}$. *Ans.* $\sqrt{8}\mp\sqrt{3}$.

2. Reduce in like manner $\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$.

Ans. $\frac{1+\sqrt{(1-x^2)}}{x}$.

3. $\left(\frac{\sqrt{(x^2+1)}-1}{\sqrt{(x^2+1)}+1}\right)^{\frac{1}{2}}=$ what?

Rendering the denominator rational by multiplying both terms by the numerator, there results

$$\left\{\frac{[\sqrt{(x^2+1)}-1]^2}{x^2}\right\}^{\frac{1}{2}}=\frac{\sqrt{(x^2+1)}-1}{x}.$$

4. Simplify the fraction $\frac{ax}{\sqrt{(a^2+x^2)}+x}$.

Ans. $\frac{x}{a}[(a^2+x^2)^{\frac{1}{2}}-x]$.

5. Given $\frac{10}{\sqrt{x}-7}=\sqrt{x}+7$, to find x. *Ans.* $x=59$.

b.) If the expression consist of more than two terms, we may proceed as follows:

$$(7^{\frac{1}{2}}+5^{\frac{1}{2}}+3^{\frac{1}{2}})(7^{\frac{1}{2}}+5^{\frac{1}{2}}-3^{\frac{1}{2}})=9+2\times 7^{\frac{1}{2}}5^{\frac{1}{2}}.$$

$$(9+2\times 7^{\frac{1}{2}}.5^{\frac{1}{2}})(9-2\times 7^{\frac{1}{2}}.5^{\frac{1}{2}})=81-4.7.5=81-140=-59$$

§ 188. *c.*) If, instead of the square root, one or both of the terms of the binomial consist of higher roots, whose numbers are powers of 2, a repetition of the process will result in a rational expression. Thus,

$$(a^{\frac{1}{4}}+b^{\frac{1}{8}})(a^{\frac{1}{4}}-b^{\frac{1}{8}})=a^{\frac{1}{2}}-b^{\frac{1}{4}};\ (a^{\frac{1}{2}}-b^{\frac{1}{4}})(a^{\frac{1}{2}}+b^{\frac{1}{4}})=a-b^{\frac{1}{2}};$$

and $$(a-b^{\frac{1}{2}})(a+b^{\frac{1}{2}})=a-b.$$

§ 189. *d.*) We have (§ 96. *d*) $a^{\frac{n}{n}}-b^{\frac{n}{n}}=(a^{\frac{1}{n}})^n-(b^{\frac{1}{n}})^n=a-b$ divisible by $a^{\frac{1}{n}}-b^{\frac{1}{n}}$.

Dividing, as in § 96. *a*, we have

$$(a-b)\div(a^{\frac{1}{n}}-b^{\frac{1}{n}})=a^{\frac{n-1}{n}}+a^{\frac{n-2}{n}}b^{\frac{1}{n}}+\ .\ .\ +a^{\frac{1}{n}}b^{\frac{n-2}{n}}+b^{\frac{n-1}{n}}.$$

$$(a^{\frac{1}{n}}-b^{\frac{1}{n}})(a^{\frac{n-1}{n}}+a^{\frac{n-2}{n}}b^{\frac{1}{n}}+.+a^{\frac{1}{n}}b^{\frac{n-2}{n}}+b^{\frac{n-1}{n}})=a-b.$$

Thus, $(a^{\frac{1}{3}}-b^{\frac{1}{3}})(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})=a-b.$

§ 190. *e.*) Again (§ 97), $a^{2n}-b^{2n}$ is divisible by $a+b$: hence $a-b$ is divisible by $a^{\frac{1}{2n}}+b^{\frac{1}{2n}}$. Dividing, we have

$$(a-b)\div(a^{\frac{1}{2n}}+b^{\frac{1}{2n}})=a^{\frac{2n-1}{2n}}-a^{\frac{2n-2}{2n}}b^{\frac{1}{2n}}+\ .\ .\ .\ -b^{\frac{2n-1}{2n}}.$$

$$\therefore\ (a^{\frac{1}{2n}}+b^{\frac{1}{2n}})(a^{\frac{2n-1}{2n}}-a^{\frac{2n-2}{2n}}b^{\frac{1}{2n}}+\ .\ .\ .\ .\ -b^{\frac{2n-1}{2n}}=a-b.$$

Thus, $(a^{\frac{1}{6}}+b^{\frac{1}{6}})(a^{\frac{5}{6}}-a^{\frac{4}{6}}b^{\frac{1}{6}}+a^{\frac{3}{6}}b^{\frac{2}{6}}-a^{\frac{2}{6}}b^{\frac{3}{6}}+a^{\frac{1}{6}}b^{\frac{4}{6}}-b^{\frac{5}{6}})=a-b.$

Also, $(5^{\frac{1}{4}}+3^{\frac{1}{4}})(5^{\frac{3}{4}}-5^{\frac{2}{4}}3^{\frac{1}{4}}+5^{\frac{1}{4}}3^{\frac{2}{4}}-3^{\frac{3}{4}})=5-3=2.$

§ 191. *f*.) Again (§ 98), $a+b$ is divisible by $a^{\frac{1}{2n+1}}+b^{\frac{1}{2n+1}}$. Thus,

$$(a+b)\div(a^{\frac{1}{2n+1}}+b^{\frac{1}{2n+1}})=a^{\frac{2n}{2n+1}}-a^{\frac{2n-1}{2n+1}}b^{\frac{1}{2n+1}}+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots -a^{\frac{1}{2n+1}}b^{\frac{2n-1}{2n+1}}+b^{\frac{2n}{2n+1}}.$$

$$\therefore\ (a^{\frac{1}{2n+1}}+b^{\frac{1}{2n+1}})(a^{\frac{2n}{2n+1}}-\ldots\ldots+b^{\frac{2n}{2n+1}})=a+b.$$

Thus, $(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})=a+b.$

So, $(7^{\frac{1}{3}}+4^{\frac{1}{3}})(7^{\frac{2}{3}}-7^{\frac{1}{3}}4^{\frac{1}{3}}+4^{\frac{2}{3}})=7+4=11.$

CHAPTER VII.

EQUATIONS OF THE SECOND DEGREE.

§ 192. We shall, at present, consider only equations, in which the *exponents* of the unknown quantities are all *integral.*

With this limitation, an equation is of the *second degree, when the difference between the highest and the lowest degrees of its terms with respect to its unknown quantity or quantities* (§ 28. *b*) *is two* (§ 40. *a*).

§ 193. *An equation containing but one unknown quantity* is, therefore, of the *second degree, when the difference between the greatest and least exponents* of the unknown quantity is *two.*

§ 194. Notes. (1.) We shall, at present, confine our attention to equations containing but one unknown quantity; and shall suppose

them to be *arranged* according to its descending powers (§ 33), and to be reduced to the *simplest form* in respect to each of those powers (§ 34. *c*). (2.) Then, each power of the unknown quantity, together with its coefficient (whether monomial or polynomial), will constitute *a term of the equation.* Thus,

Let $x^2+2ax+b^2-mx^2-4x=nx+r-q-3x^2$.

Then $(1+3-m)x^2+(2a-4-n)x+b^2+q-r=0$. §§ 33, 34. *c*, 44.

Or, making $A=1+3-m$, $B=2a-4-n$, and $C=b^2+q-r$,

$$Ax^2+Bx+C=0.$$

§ 195. An equation of the second degree, containing but one unknown quantity, its powers being all integral, may contain *any three consecutive powers, and no more.*

For, if there were more than three consecutive powers, or if there were three powers not consecutive, the difference between the greatest and least exponent must be more than two.

Thus, $Ax^3+Bx^2+Cx=0$, $Ax^2+Bx+C=0$,

$$Ax+B+Cx^{-1}(=Ax+Bx^0+Cx^{-1})=0,$$

and $Ax^{-1}+Bx^{-2}+Cx^{-3}=0$ are all of the second degree.

§ 196. Hence an equation of the second degree, when reduced as above (§ 194), can consist of only three terms (§ 194. 2); and therefore, an equation of the second degree, *consisting of three terms*, is called a *complete* equation.

§ 197. Let $Ax^3+Bx^2+Cx=0$.

Dividing by x, $Ax^2+Bx+C=0$.

Again let $Ax+B+Cx^{-1}=0$.

Dividing by x^{-1}, or multiplying by x,

$$Ax^2+Bx+C=0.$$

Or, again, let $Ax^n+Bx^{n-1}+Cx^{n-2}=0$.

Dividing by x^{n-2}, $Ax^2+Bx+C=0$. Hence,

Every complete equation of the second degree, containing only one unknown quantity, *can be reduced to the form*

$$Ax^2+Bx+C=0,$$

in which the coefficients, A, B and C, may be either *posi-*

tive or *negative*, *integral* or *fractional*, *numerical* or *algebraical*, *monomial* or *polynomial*.

Reduce the following equations to the above form.

1. $ax^2+bx+c-(mx^2+nx-p)=5x^2-8x+7$.

2. $a^2+2ar\cos v+r^2\cos^2 v+b^2+2br\sin v+r^2\sin^2 v=R^2$; r being the unknown quantity.

3. $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$.

§ 198. As the coefficients may have any value whatever, they may be equal to *zero*. But if the coefficient of a term becomes zero, the term itself becomes zero, and disappears from the equation. The equation is then sometimes called *incomplete*.

NOTES. (1.) If all the coefficients become zero at once, the equation will, of course, disappear. Also, if A and B become zero, we shall have $C=0$, and the equation will be annihilated. But, if A and C become zero, we shall have $Bx=0$, and $x=0$. Again, if B and C become zero, we shall have $Ax^2=0$, and $x=\pm 0$.

(2.) Again, let $A=0$. Then we shall have $Bx+C=0$. Now this is no longer of the second degree. It is of the first degree, and must be treated accordingly (§ 48). Neither of the above suppositions needs any further consideration.

§ 199. Now let $B=0$. Then the equation becomes

$$Ax^2+C=0.$$

$$\therefore \qquad x^2=-\frac{C}{A}=q^2 \text{ (putting } q^2=-\frac{C}{A}\text{)}.$$

$$\therefore \qquad x=\left(-\frac{C}{A}\right)^{\frac{1}{2}}=(q^2)^{\frac{1}{2}}=\pm q. \quad \text{See § 52. N.}$$

In this case, we find the values of the unknown quantity by reducing the equation to the form $x^2=q^2$, and *extracting the square root of both sides*.

Thus, let $x^2=49$.

Then $x=\sqrt{49}=\pm 7$.

NOTE. The term, *incomplete* equations of the second degree, is sometimes applied exclusively to equations of this form. They are also sometimes styled *pure* equations of the second degree, or *pure quadratics* (§ 41. N.).

a.) This form of equation will, evidently have *two roots* (§ 39) *numerically the same, but with opposite signs* (§ 23. *f.* 1).

b.) Also, if $x^2 = q^2$, then $x^2 - q^2 = 0$.

$\therefore$ $$(x+q)(x-q) = 0. \qquad \text{§ 93. III.}$$

Now it is evident, that a product will become zero, only when one of its factors is zero. The last equation, therefore, will be true, when either of its factors is equal to zero, and in no other case. Hence we may have, either

$$x+q = 0, \text{ or } x-q = 0\,;$$

and, in either case, we shall have the product

$$(x+q)(x-q) = x^2 - q^2 = 0.$$

But, if $\qquad x+q = 0$, then $x = -q$,

and if $\qquad x-q = 0$, then $x = +q$.

So $\qquad x^2 - 49 = 0$ gives $(x+7)(x-7) = 0$.

Whence,

$$x+7 = 0, \text{ and } x = -7\,; \text{ or } x-7 = 0, \text{ and } x = +7.$$

Either of these values of x will satisfy the equation, and is consequently a *root* of the equation (§ 39).

c.) If the equation, $x^2 = q^2$, or $x^2 - q^2 = 0$, be put under the complete form, thus,

$$x^2 + 0x - q^2 = 0,$$

we shall have $\qquad +q-q = 0$, the coefficient of x^1;

and $\qquad (+q)(-q) = -q^2$, the coefficient of x^0.

So, in the equation, $x^2 + 0x - 49 = 0$, we have

$$+7-7 = 0\,; \quad (+7)(-7) = -49.$$

§ 200. *d.*) We find here certain results, which will hereafter be shown to hold of all equations of the second degree, when placed under the form, $x^2 \pm 2px \pm q^2 = 0$, viz.

1. The equation can be *resolved into two binomial factors;* of which the first term of each is the unknown quantity, and the second term, with its sign changed, is a root of the equation.

2. The equation has *two roots*.

3. *The algebraic sum* of the roots, with their signs changed, is equal to the *coefficient of* x^1.

4. *The product* of the roots is equal to the *coefficient of* x^0.

Note. The student should illustrate and test these principles by applying them to the roots of every equation which he solves.

§ 201. *e.*) If the equation be of the form $x^2+q^2=0$, we shall have

$x^2=-q^2$, and, consequently, $x=\pm\sqrt{-q^2}=\pm q\sqrt{-1}$, *an imaginary result* (§§ 23. *f.* 2, 158).

Thus, let $\qquad x^2+49=0.$

Then $\quad x^2=-49$; $\therefore x=\sqrt{-49}=\pm 7\sqrt{-1}$.

Notes. (1). These expressions do not indeed represent any actual value, but they are called *roots* of the equation, because, when substituted for x, they satisfy the equation (§ 39). (2.) This imaginary result indicates an *absurdity* in the conditions of the problem. It is here proposed to find a number, whose square added to another square shall be equal to zero. That is, the sum of two positive (§ 11. N. 2) quantities is required to be zero, which is, evidently, impossible. See § 162. N. 3.

f.) The results, $x=+q\sqrt{-1}$, and $x=-q\sqrt{-1}$ give

$x-q\sqrt{-1}=0$, and $x+q\sqrt{-1}=0$; § 199. *b.*

and $\therefore$ $(x-q\sqrt{-1})(x+q\sqrt{-1})=x^2+q^2=0.$ § 200.

So $\quad (x-7\sqrt{-1})(x+7\sqrt{-1})=x^2+49=0.$

§ 202. 1. Given $5(x^2-12)=(x^2+4)$, to find x.

Ans. $x=\pm 4$.

2. Given $\dfrac{x^2-50}{2}+x=\dfrac{x^2-25}{3}+x$, to find x.

Ans. $x=\pm 10$.

3. In a right angled triangle, the square of the hypotenuse, or side opposite the right angle, is equal to the sum of the squares of the other two sides (Geom. § 188). If then the base is 4 feet, and the perpendicular 3 feet, what is the hypotenuse?

Let $x=$ the hypotenuse. Then $x^2=3^2+4^2$, &c.

4. A rope 50 feet long is extended from the top of a flag staff 40 feet high, in a straight line to the ground on the east of the flag staff, and on a level with its foot. How far from the foot of the staff will it strike the ground?

Ans. ± 30 feet (§ 5).

5. How far, if the rope be 45 feet long?

Ans. ± 20.615 &c. feet.

6. How far, if the rope be 40 feet long?

Ans. ± 0 (i. e. it will strike the ground at the foot of the staff).

7. How far, if the rope be 32 feet long?

Ans. $\pm\sqrt{-576} = \pm 24\sqrt{-1}$.

In this case, the rope, evidently, will not reach the ground; so that there is manifest absurdity in inquiring how far from the foot of the staff it will strike the ground. This absurdity is indicated by the imaginary result (§ 201. N. 2).

8. Let the perpendicular drawn from any point of the circumference of a circle to the horizontal diameter be represented by y; and let the distance from the foot of the perpendicular to the centre, measured on the horizontal diameter, be denoted by x; and the radius of the circle, by R. Then we shall have, for every point of the circumference, $x^2+y^2=R^2$; or $y^2=R^2-x^2$. § 202. 3.

Or, if the radius be 10 feet, we shall have $R^2=100$, and

$$y^2=100-x^2.$$

What now is the length of y, when $x=0$?

Ans. $y=+10$, or -10 (§ 5).

9. What is the length of y, when $x=\pm 1$, 2, 3, 4, 5, 6, 7, 8, 9, 10?

10. What is the length of y, when $x=\pm 11$?

Ans. $y=\sqrt{-21}$.

In this case, the distance measured on the horizontal diameter from the centre, being greater than the radius, extends beyond the circumference; and, of course, no perpendicular to that line at its extremity can meet the circumference. Hence the imaginary result, indicating an absurdity (§ 201. N. 2).

9. It is required to lay out 10 acres of land in a square. What must be the length of one side?

10. The product of two numbers is P, and the quotient of the greater by the less is Q. What are the numbers?

Let $x=$ the greater; then $\frac{P}{x}=$ the less; &c.

§ 203. Again, resuming the complete equation,

$$Ax^2+Bx+C=0,$$

if we suppose $C=0$, we shall have

$$Ax^2+Bx=0.$$

Dividing by x (§ 51), we have an equation of the first degree (§ 51. *b*),

$$Ax+B=0; \text{ and } \therefore x=-\frac{B}{A}.$$

a.) If, however, we divide by A, we shall have

$$x^2+\frac{B}{A}x=0, \text{ or } x^2+2px=0 \text{ (putting } 2p=\frac{B}{A}\text{)}.$$

Separating the last expression into factors, we have

$$x(x+2p)=0;$$

an equation, which will be satisfied, either when $x=0$, or when $x+2p=0$; i. e. when $x=0$, or when $x=-2p$.

The roots, therefore, of this equation regarded as of the second degree, are 0, and $-2p$ (§ 200. 2).

b.) In this case also, the *sum* of the roots with their signs changed is equal to the coefficient of x^1, and their *product*, to the coefficient of x^0 (§ 200. 3, 4). The two binomial factors (§ 200. 1) are $x-0$ and $x+2p$.

NOTE. This form of equation is frequently classed and solved as a *complete* equation of the second degree (§ 206).

1. Given $2Rx-x^2=0$, to find x.

Ans. $x=0$, or $2R$.

2. Given $r^2-2R\cos v\, r=0$, to find r.

Ans. $r=0$, or $2R\cos v$.

3. Given $x^2-10x=0$, to find x.

§ 204. Returning now to the complete equation,

$$Ax^2+Bx+C=0,$$

and dividing by A, we have $x^2+\frac{B}{A}x+\frac{C}{A}=0$;

or, putting $\qquad 2p=\frac{B}{A}$, and $q^2=\frac{C}{A}$,

$$x^2+2px+q^2=0.$$

a.) This is a *complete* equation of the second degree (§ 196); and it is perfectly general, since every complete equation can be reduced to this form by dividing by the coefficient of x^2, and substituting convenient symbols for the coefficients of x^1, and x^0.

b.) This is also the form, to which the principles of § 200 apply, and will, therefore, be commonly employed in our future discussion of the subject.

§ 205. In solving the complete equation,

$$x^2+2px+q^2=0,$$

we may happen to have $q=p$. In this case, the equation becomes

$$x^2+2px+p^2=0,$$

or (§ 93. I.), $\qquad (x+p)(x+p)=0.$

We have here the equation resolved into two binomial factors (§ 200. 1), either of which may be equal to zero. But in this case, the factors are equal; and, consequently, the values of x, found from them, will be equal. The equation is said, in this case, to have *equal* roots, viz. $-p$ and $-p$.

Thus, let $\quad x^2+20x+100=0.$

Then $\qquad (x+10)(x+10)=0$, and $x=-10$, or -10.

If we had $\quad x^2-20x+100=0$,

we should have $\qquad x=+10$, or $+10$.

a.) The sum of the roots, with their signs changed is still equal to the coefficient of x^1, and their product, to the coefficient of x^0.

§ 206. But suppose that q is not equal to p; i. e. that q^2, the coefficient of x^0, is not equal to p^2, the square of half the coefficient of x^1.

Then the equation, $x^2+2px+q^2=0$,

gives $$x^2+2px=-q^2.$$

If now we add p^2 to both sides of the equation, the first member will, evidently, become a trinomial perfect square (§§ 89, 172), and we shall have

$$x^2+2px+p^2=p^2-q^2.$$

$\therefore$ $$x+p=\surd(p^2-q^2); \qquad \text{§ 52. N.}$$

and $x=-p+\surd(p^2-q^2)$, or $x=-p-\surd(p^2-q^2)$.

Thus, let $$x^2+8x+15=0.$$

Then $$x^2+8x=-15.$$

Adding $4^2(=p^2)$, $$x^2+8x+16=-15+16=1.$$

Extracting the root, $$x+4=\pm 1.$$

$\therefore$ $$x=-4\pm 1=-3, \text{ or } -5.$$

$\therefore$ $$x+3=0, \text{ or } x+5=0 \text{ (§ 199. } b);$$

and $(x+3)(x+5)=x^2+8x+15=0$ (§§ 200, 208. b).

Also $$(-3)^2+8(-3)+15=9-24+15=0;$$

and $$(-5)^2+8(-5)+15=25-40+15=0.$$

The process of rendering the first member a perfect square, is commonly called *completing the square.*

Hence we have, for solving a complete equation of the second degree, containing but one unknown quantity, the following

RULE.

§ 207. 1. *Reduce the equation to the form* $x^2\pm 2px \pm q^2=0$. *Transpose the coefficient of* x^0 *to the second member, and add the square of half the coefficient of* x^1 *to both sides.*

2. *Extract the square root of both members, and solve the equation of the first degree thus obtained.*

1. Given $x^2+4x-60=0$, to find x.

2. Given $x^2-6x+10=65$, to find x.

3. Given $3x^2-3x+9=8\frac{1}{3}$, to find x.

Ans. $x=\frac{2}{3}$, or $\frac{1}{3}$.

4. $\frac{1}{2}x^2-\frac{1}{3}x+30\frac{1}{3}=52\frac{1}{2}$, to find x.

Ans. $x=7$, or $-6\frac{1}{3}$.

§ 208. *a.*) The same effect, obviously, will be produced. if, without transposing the coefficient of x^0, or the *absolute* term as it is sometimes called, we add to both sides *a quantity, which together with that coefficient shall be equal to the square of half the coefficient of* x^1, (i. e. p^2-q^2). Thus,

$$x^2+2px+q^2+(p^2-q^2)=p^2-q^2;$$

or $$x^2+2px+p^2=p^2-q^2.$$

$\therefore$ $$x+p=\pm\surd(p^2-q^2). \qquad \text{§ 52. N.}$$

$\therefore$ $$x=-p+\surd(p^2-q^2), \text{ or } x=-p-\surd(p^2-q^2).$$

b.) These values (§§ 206, 208) give the equations

$$x+p-(p^2-q^2)^{\frac{1}{2}}=0, \text{ and } x+p+(p^2-q^2)^{\frac{1}{2}}=0.$$

$\therefore$ $$[x+p-(p^2-q^2)^{\frac{1}{2}}][x+p+(p^2-q^2)^{\frac{1}{2}}]=$$

$$(x+p)^2-[(p^2-q^2)^{\frac{1}{2}}]^2 \text{ (§ 92) }=x^2+2px+p^2-p^2+q^2=$$

$$x^2+2px+q^2=0. \qquad \text{§ 200. 1.}$$

Also $$p-(p^2-q^2)^{\frac{1}{2}}+p+(p^2-q^2)^{\frac{1}{2}}=2p; \qquad \text{§ 200. 3.}$$

and $$[-p-(p^2-q^2)^{\frac{1}{2}}][-p+(p^2-q^2)^{\frac{1}{2}}]=q^2. \qquad \text{§ 200. 4.}$$

1. Given $x^2+6x+8=0$, to find x.

Here $q^2=8$, $2p=6$; $\therefore p=3$, $p^2=9$, and $p^2-q^2=1$.

Hence we have $x^2+6x+9=1$; and $x+3=\pm1$.

$\therefore$ $$x=-3+1=-2, \text{ or } x=-3-1=-4.$$

$\therefore$ $$x-(-2)=x+2=0, \text{ or } x-(-4)=x+4=0.$$

Hence $$(x+2)(x+4)=x^2+6x+8=0. \qquad \text{§ 200. 1.}$$

Also $$2+4=6=2p, \text{ and } 2\times4=8=q^2. \qquad \text{§ 200. 3, 4.}$$

2. Given $x^2-6x-40=0$, to find x.

Ans. $x=10$, or -4.

Here $$p^2-q^2=9-(-40)=49.$$

3. Given $x^2-16x+63=0$, to find x.

4. Given $x^2+16x+63=0$, to find x.

§ 209. *c.*) Resume the equation $Ax^2+Bx+C=0$;

or $$Ax^2+Bx=-C.$$

Multiplying by A, $A^2x^2+ABx=-AC$.

Adding $(\frac{1}{2}B)^2$, $A^2x^2+ABx+\frac{1}{4}B^2=\frac{1}{4}B^2-AC$.

Extracting the root, $Ax+\frac{1}{2}B=(\frac{1}{4}B^2-AC)^{\frac{1}{2}}$.

Hence, to complete the square,

Reduce the equation to the form $Ax^2+Bx+C=0$; *transpose the coefficient of* x^0; *multiply by the coefficient of* x^2; *and then add to both sides the square of half the primitive coefficient of* x^1.

1. Given $5x^2+4x-204=0$, to find x.

$$5x^2+4x=204.$$

$$25x^2+20x+4=1024.$$

$\therefore$ $5x+2=\pm 32$; and $\therefore$ $5x=30$, or -34.

$\therefore$ $x=6$, or $-6\frac{4}{5}$.

2. Given $2x^2+8x-90=0$, to find x.

Ans. $x=5$, or -9.

d.) Or, $$Ax^2+Bx+C=0.$$

Multiplying by A, $A^2x^2+ABx+AC=0$.

Adding $\frac{1}{4}B^2-AC$, $A^2x^2+ABx+\frac{1}{4}B^2=\frac{1}{4}B^2-AC$.

Hence, to complete the square,

Multiply the equation, $Ax^2+Bx+C=0$, *by* A, *and add to both sides* $\frac{1}{4}B^2-AC$.

Given $3x^2+2x-85=0$, to find x.

$$9x^2+6x-255=0.$$

Here $\frac{1}{4}B^2-AC=1-(-255)=256$.

$\therefore$ $9x^2+6x+1=256$.

$\therefore$ $3x+1=\pm 16$; and $\therefore$ $3x=-1\pm 16=15$, or -17.

$\therefore$ $x=5$, or $-5\frac{2}{3}$.

NOTE. When $A=1$, this solution (§ 209. *c*, *d*) is, evidently, the same as that of §§ 207, 208.

§ 210. *e.*) Again,

$$Ax^2+Bx+C=0; \text{ or } Ax^2+Bx=-C.$$

Multiplying by $4A$,

$$4A^2x^2+4ABx=-4AC.$$

Adding B^2, $\quad 4A^2x^2+4ABx+B^2=B^2-4AC.$

$$\therefore \quad 2Ax+B=(B^2-4AC)^{\frac{1}{2}}.$$

$$\therefore \quad x=\frac{-B\pm\surd(B^2-4AC)}{2A}.$$

Hence, to complete the square,

Reduce the equation to the form $Ax^2+Bx+C=0$; transpose the coefficient of x^0; multiply by four times the coefficient of x^2; and add to both sides the square of the primitive coefficient of x^1.

1. Given $3x^2-3x+\frac{2}{3}=0$, to find x.

$$36x^2-36x=-8.$$

$$36x^2-36x+9=-8+9=1.$$

$$\therefore \quad 6x-3=\pm1; \therefore 6x=3\pm1=4, \text{ or } 2.$$

$$\therefore \quad x=\tfrac{2}{3}, \text{ or } \tfrac{1}{3}.$$

2. Given $\frac{1}{2}x^2-\frac{1}{3}x-22\frac{1}{6}=0$, to find x.

Ans. $x=7$, or $-6\frac{1}{3}$.

f.) Or, multiply the equation, $Ax^2+Bx+C=0$, by $4A$.

Then $\quad 4A^2x^2+4ABx+4AC=0.$

Adding B^2-4AC,

$$4A^2x^2+4ABx+B^2=B^2-4AC; \text{ as in } e \text{ above.}$$

Hence, to complete the square,

Multiply the equation, $Ax^2+Bx+C=0$, by $4A$; and add to both sides B^2-4AC.

Given $\quad x^2-5x-24=0$, to find x.

$$4x^2-20x-96=0.$$

Here $\quad B^2-4AC=25-(-96)=25+96=121.$

$$\therefore \quad 4x^2-20x+25=121.$$

$$\therefore \quad 2x-5=\pm11; \therefore x=8, \text{ or } -3.$$

NOTE. When $A=\frac{1}{2}$, this solution (§ 210. *e*, *f*) is, obviously, the same as that of §§ 207, 208.

§ 211. Let $x^2+2px+q^2=0$ be any equation whatever of the second degree, containing but one unknown quantity; let also a be one of its roots (i. e. such a quantity as, being substituted for x in the given equation, will make the members equal; or, in other words, will reduce the first member to zero). See § 39.

1. Since a is a root of the equation, we have $x=a$, and $x-a=0$.

Divide the given equation by $x-a$.

Thus
$$\begin{array}{l|l} x^2+2px+q^2 & x-a \\ \underline{x^2-ax} & \overline{x+(a+2p)} \\ (a+2p)x & \\ \underline{(a+2p)x-a^2-2pa} & \\ \qquad a^2+2pa+q^2=0, & \end{array}$$

because the remainder is simply the first member of the given equation with a substituted for x; which, by hypothesis, reduces it to zero. The division is therefore perfect (§ 82. g).

Hence $(x-a)(x+a+2p)=x^2+2px+q^2=0$. § 200. 1.

2. And the equation will be satisfied, if we take $x-a=0$, or $x+2p+a=0$ (i. e. if $x=a$, or $x=-2p-a=b$ (by substitution). § 200. 2.

3. We have also $-a+(2p+a)=2p$. § 200. 3.

4. Moreover, since $a^2+2pa+q^2=0$, (see 1, above),

$$a(-a-2p)[=-a^2-2pa]=q^2. \qquad \S\ 200.\ 4.$$

§ 212. 5. It is also evident from § 211. 1, that, if a is a root of the equation $x^2+2px+q^2=0$, this equation is *divisible by $x-a$*, and will give a quotient of the form $x-b$, of which the second term is the other root with its sign changed.

§ 213. Hence, universally (§ 200),

1. Every equation of the second degree, of the form $x^2\pm 2px\pm q^2=0$, containing but one unknown quantity, can be *resolved into two binomial factors*, of the first degree

in respect to x (§ 28. *b*); either of which, being put equal to zero, gives a root of the equation.

2. Every such equation has, of course, *two roots.*

3. *The algebraic sum* of the roots, with their signs changed, is always equal to the coefficient of x^1.

4. *The product* of the roots is always equal to the *coefficient of* x^0.

5. Every such equation, of which a is a root, is *divisible* by $x-a$.

§ 214. *a.*) Hence (§ 213. 3, 4),

Cor. I. (1.). *If the coefficient of* x^1 *be equal to zero, the roots must be numerically the same, but with opposite signs* (§ 199. *a*). (2.) *If the coefficient of* x^0 *be equal to zero, one of the roots must be zero* (§ 203. *a*).

§ 215. *b.*) Also (§§ 213. 4; 9. *a*; 213. 3),

Cor. II. (1.) *If the coefficient of* x^0 *be* POSITIVE, *the roots must have* LIKE *signs;* (2.) *if* NEGATIVE, UNLIKE. (3.) *If the two roots have the same sign, it will be unlike the sign of the coefficient of* x^1. (4.) *If they have different signs, the sign of the root which is numerically the greater will be unlike that of the coefficient of* x^1.

c.) It is obvious, that, if the roots have like signs, the coefficient of x^1 will be numerically equal to their arithmetical sum; and, if they have unlike signs, to their arithmetical difference.

§ 216. *d.*) If q^2 be positive and greater than p^2, the product of two numbers is required to be greater than the square of half their sum. This will be shown to be impossible (§ 220. *b*), and as no real numbers can satisfy this condition, the roots will be imaginary (§§ 201; 217. I.). Hence,

Cor. III. *If the coefficient of* x^0 *be positive and greater than the square of half the coefficient of* x^1, *the roots must be imaginary.*

§ 217. *e.*) The above principles may be otherwise demonstrated; thus,

I. Let q^2 be *positive.*

Then $x^2 \pm 2px + q^2 = 0$; and $x = \mp p \pm \surd(p^2 \mp q^2)$.

Now, evidently, $\surd(p^2 - q^2) < p$; and, therefore, both the roots are *negative*, when $2p$ is *positive;* and *positive*, when $2p$ is *negative* (§ 215. 1, 3).

It is also evident, that, if $q^2 > p^2$, $\surd(p^2 - q^2$ is *imaginary* (§§ 23. *f.* 2; 216.).

It is also manifest, that, if one of the roots is imaginary, both must be.

II. Again, let q^2 be *negative.* Then $x^2 \pm 2px - q^2 = 0$; and $x = \mp p \pm \surd(p^2 + q^2)$.

Here, obviously, $\surd(p^2 + q^2) > p$; and, therefore, one root must be of the same sign as p; and the other, different (§ 215. 2).

Also, the root which is of the same sign as p (i. e. of a sign different from $2p$ on the other side), will, of course, be numerically the greater (§ 215. 4).

§ 218. *f.*) Determine whether the signs of the roots in the following equations are like or unlike; if like, whether positive or negative; and, if unlike, which is numerically the greater. Also determine whether any of these equations have imaginary roots.

1. $x^2 + 21x + 110 = 0$; $x^2 - 20 + 75 = 0$.
2. $x^2 - 23x + 130 = 0$; $x^2 + 23x + 130 = 0$.
3. $x^2 \pm 60x + 1000 = 0$; $x^2 \pm 60x - 1000 = 0$.
4. $x^2 \pm 60x - 11200 = 0$; $x^2 \pm 10x = 200$.

g.) 1. Write the equation, of which 3 and 4 are the roots.

Ans. $(x-3)(x-4) = x^2 - 7x + 12 = 0$.

2. Write the equation, whose roots are -3 and -4; -11 and $+20$; $+11$ and -20; -10 and $+10$; -10 and -10; $10 + \surd{-5}$ and $10 - \surd{-5}$; $-6 + 5\surd{-1}$ and $-6 - 5\surd{-1}$.

h.) In the last example, we have (§§ 92, 162)

$(x+6-5\surd-1)(x+6+5\surd-1)=(x+6)^2-(5\surd-1)^2=(x+6)^2+5^2=0$; which is, evidently, impossible (§ 201. N. 2).

§ 219. *i.*) Again (§ 210), we shall have the value,

$x=\dfrac{-B\pm\surd(B^2-4AC)}{2A}$, *real*, when B^2-4AC is *positive;* and *imaginary*, when B^2-4AC is *negative.* That is, the roots will be *real and unequal*, when $B^2-4AC>0$;

real and equal, " $B^2-4AC=0$;

imaginary, " $B^2-4AC<0$.

PROBLEMS.

§ 220. 1. Given $x^2-2x-24=0$, to find the values of x. *Ans.* $x=+6$, and -4.

2. Given $x^2+12x+35=0$, to find x.
Ans. $x=-5$, or -7.

3. Given $3x^2+2x-10=75$, to find x.
Ans. $x=5$, or $-5\frac{2}{3}$.

4. Given $x^2-x-210=0$, to find x.
Ans. $x=15$, or -14.

5. Given $\frac{1}{2}x^2-\frac{1}{3}x+6\frac{3}{8}=7$, to find x.
Ans. $x=1\frac{1}{2}$, or $-\frac{5}{6}$.

6. Find two numbers whose sum is 100, and whose product is 2100.

Let $x=$ one of the numbers.

Then $100-x=$ the other;

and $x(100-x)=2100$, by the second condition.

$\therefore$ $x^2-100x=-2100$.

We might have formed this equation immediately by considering, that the sum of the required numbers taken with a contrary sign must be equal to the coefficient of x^1: and their product, to the coefficient of x^0.

Thus $x^2-100x+2100=0$.

$\therefore \qquad x^2-100x+2500=400.$ § 208.

$\therefore \qquad x=70$, or 30.

Otherwise, let $x=$ the excess of the greater number above 50 (i. e. half the sum of the numbers); then $50+x=$ the greater, and $50-x=$ the less.

Hence $\quad [(50+x)(50-x)=]2500-x^2=2100.$

$\therefore \qquad x^2=400$; and $x=\pm 20$.

$\therefore \qquad 50+x=70$, or 30; and $50-x=30$, or 70.

7. Find two numbers, whose sum is 100, and whose product is 2400.

8. Find two numbers, whose sum is 100, and whose product is 2500 (§ 205).

9. Find two numbers, whose sum is 100, and whose product is 2600 (§ 216).

Ans. $50+10\sqrt{-1}$, and $50-10\sqrt{-1}$.

10. Find two numbers, whose sum is S, and product P.

Ans. $\dfrac{S}{2}+\left(\dfrac{S^2}{4}-P\right)^{\frac{1}{2}}$, and $\dfrac{S}{2}-\left(\dfrac{S^2}{4}-P\right)^{\frac{1}{2}}$.

a.) In what case will these values be imaginary?

Ans. When $P>\dfrac{S^2}{4}\left[=\left(\dfrac{S}{2}\right)^2\right]$. See 9, above.

Hence,

The product of two numbers can never be greater than the square of half their sum.

b.) This principle can be proved otherwise; thus,

Let S be the sum of two numbers, and D, their difference.

Then $\qquad \frac{1}{2}S+\frac{1}{2}D=$ the greater, § 57. 3.

and $\qquad \frac{1}{2}S-\frac{1}{2}D=$ the less. § 60. 4.

Also $(\frac{1}{2}S+\frac{1}{2}D)(\frac{1}{2}S-\frac{1}{2}D)=(\frac{1}{2}S)^2-(\frac{1}{2}D)^2=$ their product; which is obviously greatest when $(\frac{1}{2}D)^2$ is least, i. e. when $\frac{1}{2}D=0$.

11. The algebraic sum of two numbers is 8, and their product is -240. What are the numbers?

Here, by § 200, 3, 4,

$$x^2-8x-240=0, \text{ or } x^2-8x=240.$$

$$\therefore \qquad x=20, \text{ or } -12.$$

Verification. If one of the numbers is 20, the other is $8-20=-12$; and $20\times-12=-240$. Or, if one of the numbers is -12, the other is $8-(-12)=20$, &c. Or, if one of the numbers is 20, the other is $-240\div20=-12$, and $-12+20=8$.

12. Find two numbers, whose difference is 10; and such that, if 600 be divided by each of them, the difference of their quotients shall be 10.

13. Find a number, which added to its square makes 42. *Ans.* 6, or -7.

14. Find two numbers, whose sum is 16, and the sum of whose squares is 130. *Ans.* 7 and 9.

15. What two numbers are there, whose sum is S, and the sum of whose squares is Q?

Ans. $\frac{1}{2}S+\frac{1}{2}\surd(2Q-S^2)$, and $\frac{1}{2}S-\frac{1}{2}\surd(2Q-S^2)$.

c.) When will these results be imaginary?

Ans. When $S^2>2Q$. Whence,

The square of the sum of two numbers cannot be greater than twice the sum of their squares.

NOTE. As either of the numbers may be negative, this applies equally to the square of the difference.

16. The sum of two numbers is 25, and the sum of their cubes is 8125. What are the numbers? *Ans.* 20 and 5.

17. A rectangular field contains 20 acres, and one side is 40 rods longer than the other. What are the dimensions of the field? *Ans.* 80 rods long, and 40 wide.

18. A rectangular park, 60 rods long and 40 wide, is surrounded by a street of uniform width, containing 1344 square rods. How wide is the street?

Ans. 6 rods, or -56 rods.

d.) The second value, -56, is clearly not a proper solution to the problem; but it is a root of the equation, and,

in some sense, satisfies the conditions of the problem. For we find the area of the street by multiplying its width by each of the sides of the park, and adding to the sum of these products the squares formed at the four corners. Thus,

$$6\times60+6\times60+6\times40+6\times40+4\times6^2=1344.$$

So $$2(-56\times60)+2(-56\times40)+4(-56)^2=1344.$$

It frequently happens, as we have already seen (§ 137), that the algebraic expression of a problem is more general, and admits of more solutions, than the problem itself as expressed in ordinary language.

$$x^{2n}+Px^n+Q=0.$$

§ 221. The preceding methods apply not only to equations of the second degree, but to all equations of the form

$$x^{2n}+Px^n+Q=0,$$

in which the unknown quantity appears in only two terms; and its exponent in one of the terms is double that in the other.

This equation may be put under the form,

$$(x^n)^2+Px^n+Q=0.$$

Completing the square (§ 207),

$$(x^n)^2+Px^n+\tfrac{1}{4}P^2=\tfrac{1}{4}P^2-Q.$$

$$\therefore \qquad x^n=-\tfrac{1}{2}P\pm(\tfrac{1}{4}P^2-Q)^{\frac{1}{2}}.$$

$$\therefore \qquad x=[-\tfrac{1}{2}P\pm(\tfrac{1}{4}P^2-Q)^{\frac{1}{2}}]^{\frac{1}{n}}. \qquad \text{§ 52. N.}$$

1. Given $x^4-52x^2+576=0$, to find x.

Ans. $x=\pm6$, or ±4.

2. Given $\frac{1}{3}x-\frac{1}{2}\surd x=1\frac{1}{2}$, to find x.

Ans. $\surd x=3$, or $-1\frac{1}{2}$; $\therefore x=9$, or $2\frac{1}{4}$.

In verifying these results, $\surd x$ must be positive for the first value, and negative, for the second. A similar remark applies to the following example.

3. Given $(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}=6$, to find x.

$$(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}+\tfrac{1}{4}=\tfrac{1}{4}+6=\frac{25}{4}.$$

$$\therefore \qquad (x+12)^{\frac{1}{4}} = -\tfrac{1}{2} \pm \tfrac{5}{2} = 2, \text{ or } -3.$$

$$\therefore \qquad x+12 = 16, \text{ or } 81. \qquad \S\, 52.$$

$$\therefore \qquad x = 4, \text{ or } 69.$$

4. Given $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$, to find x.

Ans. $x = 243$, or $(-28)^{\frac{5}{3}}$.

5. Given $x^3 - x^{\frac{3}{2}} = 56$, to find x.

Ans. $x = 4$, or $(-7)^{\frac{2}{3}}$.

NOTE. We have seen (§ 213. 2) that every equation of the second degree has two roots. It will be proved hereafter, that *every equation has as many roots as there are units in its degree.* See 1, above. The above process, however, does not always exhibit all the roots.

§ 222. If an equation contain *radicals* which cannot be treated by the method of § 221, it may frequently be reduced by properly arranging the radical terms containing the unknown quantity, and raising both members to the requisite power. There is frequently great advantage also in rendering a binomial surd rational (§§ 186, 187).

The radicals, which most frequently occur, are *radicals of the second degree* (i. e. expressions of the *square* root of quantities).

1. Given $x + \sqrt{(2ax + x^2)} = a$, to find x.

We have $\sqrt{(2ax + x^2)} = a - x$.

Then squaring $2ax + x^2 = a^2 - 2ax + x^2$.

$\therefore \qquad 4ax = a^2$; and $x = \frac{1}{4}a$.

2. Given $\left(\frac{x+a}{x}\right)^{\frac{1}{2}} + 2\left(\frac{a}{x+a}\right)^{\frac{1}{2}} = b^2\left(\frac{x}{x+a}\right)^{\frac{1}{2}}$, to find x.

Clearing of fractions, $x + a + 2\sqrt{(ax)} = b^2 x$.

Extracting the square root,

$$\sqrt{x} + \sqrt{a} = \pm b\sqrt{x};$$

or

$$(1 \mp b)\sqrt{x} = -\sqrt{a}.$$

$$\therefore \qquad (1 \mp b)^2 x = a.$$

$$\therefore \qquad x = \frac{a}{(1 \mp b)^2} = \frac{a}{(b \mp 1)^2}.$$

3. Given $2x+2\sqrt{(a^2+x^2)}=\frac{5a^2}{\sqrt{(a^2+x^2)}}$, to find x.

Ans. $x=\frac{3}{4}a$.

4. Given $\frac{\sqrt{x}+\sqrt{(x-a)}}{\sqrt{x}-\sqrt{(x-a)}}=\frac{n^2a}{x-a}$, to find x.

If we render the denominator of the first member rational (§ 187), multiply by a, and extract the square root, we shall have

$$\sqrt{x}+\sqrt{(x-a)}=\frac{\pm na}{\sqrt{(x-a)}}.$$

Clearing of fractions and transposing,

$$\sqrt{(x^2-ax)}=a\pm na-x=(1\pm n)a-x.$$

Squaring, $\quad x^2-ax=(1\pm n)^2a^2-2(1\pm n)ax+x^2.$

$$\therefore \qquad x=\frac{(1\pm n)^2a}{1\pm 2n}.$$

5. Given $\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=b$, to find x.

Ans. $x=\frac{2ab}{b^2+1}$.

6. Given $\frac{\sqrt{(4x+1)}+\sqrt{(4x)}}{\sqrt{(4x+1)}-\sqrt{(4x)}}=9$, to find x.

Ans. $x=\frac{4}{5}$.

7. Given $\frac{1}{1-\sqrt{(1-x^2)}}-\frac{1}{1+\sqrt{(1-x^2)}}=\frac{\sqrt{3}}{x^2}$, to find the value of x. *Ans.* $x=\pm\frac{1}{2}$.

§ 223. Every complete equation of the second degree, containing two unknown quantities, and having only positive integral powers (§ 22. *c*, *d*), is, obviously, of the form (§ 197)

$$Ay^2+Bxy+Cx^2+Dy+Ex+F=0.$$

That is, it contains terms of the zero, the first, and the second degree with respect to both and each of the unknown quantities.

a.) A single equation of this kind is, of course, indeterminate (§ 122. *a*); and will give, for any value whatever of

either of the unknown quantities, two values of the other (§ 213. 2).

§ 224. *b.*) Such equations are of continual use in the higher applications of Algebra, in expressing the relation between two *variable* (§ 136) quantities which are so connected, that a change in the value of one, in general, involves a change in the value of the other; i. e. between two *variables*, which are *functions*, one of the other (§§ 26. 136. *a*).

c.) Thus, let x denote the distance from any point in the circumference of a circle to a given straight line, and y the distance from the same point to another line perpendicular to the first. Then the relation between these distances will be such, that, if one of them be given, the other will be determined; and if another point be taken at a different distance from the first line, it will also, in general, be at a different distance from the second. That is, a particular value of x requires a corresponding value of y; and a change in the value of x involves, in general, a corresponding change in the value of y.

d.) An equation, expressing some known relation between these distances, is called an *equation of the curve.* By means of such an equation, the properties of the curve are easily and rapidly deduced.

e.) The equations of the circle, ellipse, parabola and hyperbola are of the second degree, and contain two variables.

Thus, $y^2+x^2-R^2=0$ is the equation of the circumference of a circle, when the distances x and y are measured from two diameters at right angles to each other. For in that case these distances for any point of the curve, together with the radius drawn to that point, form a right angled triangle, of which the radius is the hypotenuse. Whence $x^2+y^2=R^2$ (Geom. § 188).

NOTE. A straight line is represented by an equation of the *first* degree, between two variables. Thus, $y=ax+b$; a and b being either positive or negative.

f.) The employment of equations of this kind for the discovery of geometrical truth belongs to Analytical Geometry and to the Differential and Integral Calculus. And this, at the same time, furnishes one of the most important applications of the principles, already demonstrated, of equations of the second degree.

§ 225. The ordinary algebraic treatment of equations of the second degree, containing two unknown quantities, supposes two equations (§ 122. *d*, *e*); and deduces values for the unknown quantities, which will satisfy both equations.

Let there be given the two equations,

$$Ay^2+Byx+Cx^2+Dy+Ex+F=0,$$

and $$A'y^2+B'yx+C'x^2+D'y+E'x+F'=0.$$

If now one of the unknown quantities, as y, be found in terms of x and known quantities, and this value be substituted in the other equation, there will, of course, result an equation containing but one unknown quantity. If this equation be solved, and the values found for x be substituted in one of the primitive equations, corresponding values of y may be found.

But it is sufficiently evident, that the equation so obtained by the elimination of one of the unknown quantities will be of the fourth degree, which, in its general form, we are not yet prepared to solve.

§ 226. Though we are not prepared for a general solution of two equations of the second degree containing two unknown quantities, yet certain classes of such equations, can be solved by applying the principles already demonstrated.

This is true of all those equations, in which the elimination of one of the unknown quantities results in an equation either of the second degree, or of the form, $x^{2n}\pm Px^n \pm Q=0$ (§ 221).

1. Given $x^2+y^2=100$,

$x^2-y^2=28$, to find x and y.

Adding, subtracting, and dividing by 2, we have

$$x^2=64, \therefore x=\pm 8;$$

and $$y^2=36, \therefore y=\pm 6.$$

2. Given $x^2+y^2=100$,

$xy=48$, to find x and y.

From the second equation,

$$y^2=\left(\frac{48}{x}\right)^2.$$

Substituting in the first,

$$x^2+\frac{2304}{x^2}=100.$$

$\therefore$ $$x^4-100x^2=-2304. \qquad \S\ 221.$$

$\therefore$ $$x^2=64, \text{ or } 36; \text{ and } x=\pm 8, \text{ or } \pm 6.$$

$\therefore$ $$y=\pm 6, \text{ or } \pm 8.$$

a.) The last example may be more conveniently solved without elimination. Thus, adding and subtracting twice the second equation to and from the first, we have

$$x^2+2xy+y^2=196;$$

and $$x^2-2xy+y^2=4.$$

$\therefore$ $$x+y=\pm 14; \text{ and } x-y=\pm 2.$$

$\therefore$ $x=\pm 8$, or ± 6; and $y=\pm 6$, or ± 8; as before.

3. Given $x^2+xy+y^2=112$,

$x^2-xy+y^2=48$, to find x and y.

Ans. $x=\pm 8$, $y=\pm 4$.

4. Given $x^2+xy=180$,

$xy+y^2=45$, to find x and y.

Ans. $x=\pm 12$, $y=\pm 3$.

5. Given $4xy=96-x^2y^2$,

$x+y=6$, to find x and y.

Ans. $x=4$, or 2, or $3\pm\sqrt{21}$

$y=2$, or 4, or $3\mp\sqrt{21}$.

6. Given $x^2+x+y=18-y^2$,

$xy=6$, to find x and y.

Ans. $x=3$, or 2, or $-3\pm\sqrt{3}$,

$y=2$, or 3, or $-3\mp\sqrt{3}$.

§ 227. *b.*) If one of the equations be of the first degree, and one of the quantities be eliminated, then the resulting equation will be only of the second degree.

1. Given $2x+y=10$,
$2x^2-xy+3y^2=54$, to find x and y.

From the first, $y=10-2x$.

Substituting, $2x^2-x(10-2x)+3(10-2x)^2=54$.

Ans. $x=3$, or $5\frac{1}{8}$; $y=4$, or $-\frac{1}{4}$.

2. Given $2x+y=9$,
$xy=10$, to find x and y.

Ans. $x=2$, or $2\frac{1}{2}$; $y=5$, or 4.

3. Given $x+y=10$,
$x^2+y^2=50$, to find x and y.

§ 228. *c.*) It is sometimes convenient to employ *auxiliary* unknown quantities, such as the sum and difference, or the sum or difference and product or quotient.

NOTE. If one or both of the equations be of a *higher* degree, the problem can frequently be solved by an equation of the second degree.

1. Given $x+y=a$,
$x^3+y^3=b$, to find x and y.

Let $x=s+t$, $y=s-t$, and $\therefore$ (§ 57. 3) $s=\frac{1}{2}(x+y)=\frac{1}{2}a$.

Then $x^3+y^3=(s+t)^3+(s-t)^3=2s^3+6st^2=b$.

Hence $t^2=\dfrac{b-2s^3}{6s}$, and $t=\pm\left(\dfrac{b-2s^3}{6s}\right)^{\frac{1}{2}}$.

$\therefore$ $x=s\pm\left(\dfrac{b-2s^3}{6s}\right)^{\frac{1}{2}}$; and $y=s\mp\left(\dfrac{b-2s^3}{6s}\right)^{\frac{1}{2}}$.

$\therefore$, introducing the value of s,

$$x=\frac{a}{2}\pm\left(\frac{b-\frac{1}{4}a^3}{3a}\right)^{\frac{1}{2}}=\frac{a}{2}\pm\left(\frac{4b-a^3}{12a}\right)^{\frac{1}{2}};$$

and $$y=\frac{a}{2}\mp\left(\frac{4b-a^3}{12a}\right)^{\frac{1}{2}}.$$

Let $a=10$, and $b=370$; $a=12$, and $b=1008$; $a=7$, and $b=217$.

2. Given $x+y=8$,

$$x^4+y^4=706, \text{ to find } x \text{ and } y.$$

Let $x=s+t$, $y=s-t$, and $\therefore$ $s=\frac{1}{2}(x+y)=4$.

Then $x^4+y^4=(s+t)^4+(s-t)^4=2s^4+12s^2t^2+2t^4=706$.

Or, as $s=4$, $\quad 512+192t^2+2t^4=706$.

$\therefore \quad t^4+96t^2=97$.

$\therefore \quad t^2=1$, or -97; and $t=\pm 1$, or $\pm\sqrt{-97}$.

$\therefore \quad x=5$, and $y=3$; or $x=3$, $y=5$;

or $\quad x=4\pm\sqrt{-97}$, and $y=4\mp\sqrt{-97}$.

3. Given $4x^2-2xy=12$,

$$2y^2+3xy=8, \text{ to find } x \text{ and } y.$$

Assume $x=zy$, i. e. substitute zy for x.

Ans. $x=\pm 2$, or $\pm\frac{3}{7}\sqrt{7}$.
$y=\pm 1$, or $\mp\frac{8}{7}\sqrt{7}$.

4. Given $3x^2+xy=68$,

$$4y^2+3xy=160, \text{ to find } x \text{ and } y.$$

Ans. $x=\pm 4$, or $\mp\frac{34}{9}\sqrt{3}$,
$y=\pm 5$, or $\pm\frac{16}{3}\sqrt{3}$.

d.) Frequently, by a little *reduction*, the form of the equations can be changed, so as to be conveniently solved.

5. Given $x^3y-y=21$

$$x^2y-xy=6, \text{ to find } x \text{ and } y.$$

Finding y from each, and equating the two values, the two sides of the equations will be found to have a common factor.

6. Given $x^2+3x+y=73-2xy$,

$$y^2+3y+x=44, \text{ to find } x \text{ and } y.$$

If we add these equations and transpose, there will result an equation, from which $x+y$ can be found.

CHAPTER VIII.

RATIO AND PROPORTION.

§ 229. In considering the relative magnitude of quantities of the same kind, we may inquire, either *how much* one *exceeds* the other, or *how many times* the one *contains* the other. The former of these relations is simply the *difference* of the quantities; the latter, their *quotient*, is also called their RATIO.[k]

§ 230. The RATIO of two quantities is the *relation* expressed by *dividing one of the quantities by the other.*

Thus, the ratio of 2 to 3 is $\frac{2}{3}$ (otherwise written $2:3$); that of a to b is $\frac{a}{b}$ (otherwise, $a:b$).

a.) These are merely different ways of expressing the same thing; a ratio being simply a fraction.

b.) The first term (§ 111. N.) of a ratio is called the *antecedent*, and the second the *consequent*, of the ratio.

c.) A ratio being simply a fraction, *its terms may be* BOTH *multiplied* or BOTH *divided by the same number* without altering the value of the ratio (§ 113. 3). Thus, the ratio of 2 to 3 is the same as that of 2×5 to 3×5, or of $\frac{2}{5}$ to $\frac{3}{5}$. So the ratio of a to b is the same as that of am to bm, or of $\frac{a}{m}$ to $\frac{b}{m}$. That is,

(*k*) Lat., *relation.*

$$\frac{2}{3}=\frac{2\times5}{3\times5}=\frac{\frac{2}{5}}{\frac{3}{5}};\ \frac{a}{b}=\frac{am}{bm}=\frac{\frac{a}{m}}{\frac{b}{m}}.$$

or, $$2:3=2\times5:3\times5=\tfrac{2}{5}:\tfrac{3}{5},\ \&c.$$

§ 231. A PROPORTION *is an equation* consisting of *two equal ratios*.

Thus $2:3=6:9$, or $\frac{2}{3}=\frac{6}{9}$ is a proportion.

So, $a:b=m:n$, or $\dfrac{a}{b}=\dfrac{m}{n}$.

NOTE. Instead of the sign of equality, four dots (::) are sometimes used. Thus, we may write indifferently $2:3::6:9$, or $2:3=6:9$ (read in either case 2 is to 3 as 6 to 9). The sign of equality is, however, preferable.

a.) The terms of the ratio are called also terms of the proportion. The first and last terms are called the *extremes*, and the second and third, the *means* of the proportion.

b.) If the second and third terms be the same, that quantity is said to be a MEAN PROPORTIONAL *between the other two*. Thus,

$$2:4=4:8;\quad a^2:ab=ab:b^2;$$

or $$\tfrac{2}{4}=\tfrac{4}{8};\quad \frac{a^2}{ab}=\frac{ab}{b^2}.$$

Here 4 is a mean proportional between 2 and 8; and ab, between a^2 and b^2.

NOTE. In this case, the three terms are said to be in *continued* proportion.

§ 232. Let $a:b=k:l$, or $\dfrac{a}{b}=\dfrac{k}{l}$.

Clearing of fractions, $al=bk$. That is,

In any proportion, *the product of the extremes is equal to the product of the means.*

Thus, if $5:7=10:14$, then $5\times14=10\times7$.

Hence,

a. Cor. I.) *If any three terms of a proportion be given, the fourth may be found.*

For, if the means and one extreme be given, the other extreme will be found by dividing the product of the means by the given extreme. Or, if the extremes and one mean be given, the other mean will be found by dividing the product of the extremes by the given mean. Thus,

If $x : 6 = 11 : 22$, then $22x = 66$; and $x = 3$.

So, if $5 : x = 10 : 40$, then $10x = 200$; and $x = 20$.

Or, if $5 : 13 = 15 : x$, then $x = 39$.

NOTE. The last is the form ordinarily used in Arithmetic.

b.) Again, let $a : x = x : b$.

Then $ab = x^2$. Hence,

Cor. II. If *three* terms be in *continued* (§ 231. *b.* N.) proportion, *the product of the extremes* is equal to the *square of the mean.*

Thus, if $2 : 12 = 12 : 72$, then $2 \times 72 = 12 \times 12 = 12^2$.

c.) Also, if $a : x = x : b$, then

$$x^2 = ab; \text{ and } x = (ab)^{\frac{1}{2}}. \quad \text{Hence,}$$

Cor. III. *The mean proportional between two numbers is equal to the square root of their product.*

Thus if $3 : x = x : 48$, then $x = (3 \times 48)^{\frac{1}{2}} = 12$.

Find a mean proportional between 1 and 9; between 2 and 8; between 5 and 500; between a^2 and b^2; between $R+x$ and $R-x$.

§ 233. Let $al = bk$.

Dividing both members by b and by l,

$$\frac{a}{b} = \frac{k}{l}, \text{ or } a : b = k : l. \quad \text{Hence,}$$

If the product of two numbers be equal to the product of two other numbers, *the two factors of either product may be made the means, and the two factors of the other product the extremes of a proportion.*

Thus, if $7\times15=3\times35$,

then $7:3=35:15$, or $7:35=3:15$, &c.

So, if $x''x-x''^2[=x''(x-x'')]=A^2-x''^2$,

then $x'':A-x''=A+x'':x-x''$.

1. What proportion results from the equation $\sin(a+b)\sin(a-b)=\sin^2a-\sin^2b$?

Ans. $\sin(a-b):\sin a-\sin b=\sin a+\sin b:\sin(a+b)$.

2. What proportion from the equation $\sin b\sin C=\sin c\sin B$?

a.) Also, if $x^2=ab$, then $a:x=x:b$.

Hence, evidently,

Cor. *If the product of two numbers be equal to the square of a third, this last is a mean proportional between the other two.*

Thus, if $12^2=2\times72$, then $2:12=12:72$.

So, if $y^2=R^2-x^2$, then $R+x:y=y:R-x$.

Transform the following equations into proportions.

1. $y^2=2Rx-x^2$. *Ans.* $x:y=y:2R-x$.
2. $y^2=2px$. *Ans.* $x:y=y:2p$.
3. $R^2=\tan a\cot a$; $A^2=x''x$.

§ 234. Let $a:b=k:l$, or $\frac{a}{b}=\frac{k}{l}$.

I. Multiplying by b, and dividing by k,

$$\frac{a}{k}=\frac{b}{l};\text{ or } a:k=b:l.$$

Or, multiplying by l, and dividing by a,

$$\frac{l}{b}=\frac{k}{a};\text{ or } l:b=k:a.$$ Hence,

The means or the extremes of a proportion may exchange places.

Thus, if $2:3=8:12$, then $2:8=3:12$.

NOTE. The interchange of the means is called ALTERNATION(l);

(*l*) Lat. alterno, *to interchange;* hence alternando, *by interchanging.*

and the quantities are said to be in proportion *alternately*, or *alternando*.

§ 235. II. Again $1 \div \frac{a}{b} = 1 \div \frac{k}{l}$.

$\therefore \qquad \frac{b}{a} = \frac{l}{k}$; or $b : a = l : k$. Hence,

The terms of each ratio of a proportion may exchange places; i. e. the antecedent may be made consequent, and the consequent, antecedent.

Thus, if $2 : 3 = 8 : 12$, then $3 : 2 = 12 : 8$.

NOTE. This is called INVERSION[m]; and the quantities are said te be in proportion *by inversion*, or *invertendo*.

§ 236. III. Adding ± 1 to each side,

$$\frac{a}{b} \pm 1 = \frac{k}{l} \pm 1.$$

$\therefore$ (§ 114. *a*) $\frac{a \pm b}{b} = \frac{k \pm l}{l}$; or $a \pm b : b = k \pm l : l$. (1)

Again (§ 235) $\frac{b}{a} = \frac{l}{k}$. $\therefore 1 \pm \frac{b}{a} = 1 \pm \frac{l}{k}$.

$\therefore \qquad \frac{a \pm b}{a} = \frac{k \pm l}{k}$; or $a \pm b : a = k \pm l : k$. (2)

Hence,

The sum or difference of the first and second is to either the first or second, as the sum or difference of the third and fourth is to the third or fourth.

Thus, if $7 : 5 = 14 : 10$, then $7 \pm 5 : 7 = 14 \pm 10 : 14$.

NOTE. In this case the quantities are said to be in proportion *by composition*,[n] or *componendo*, when the sum is taken; and *by division* or *dividendo*[o], when the difference is taken.

a.) Also $\qquad a+b : k+l = a : k$; § 234.

and $\qquad a-b : k-l = a : k$.

$\therefore \qquad a+b : k+l = a-b : k-l$.

or (§ 234) $\qquad a+b : a-b = k+l : k-l$.

(*m*) Lat. inverto, *to invert;* hence invertendo, *by inverting*. (*n*) Lat. compono, *to compound*, hence componendo, *by compounding*. (*o*) Lat., from divido, *to separate; by separating*.

Hence,

Cor. *The sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.*

Thus $3:2=6:4$; $\therefore 3+2:3-2=6+4:6-4$.

Hence (§§ 234–236),

§ 237. *If four quantities be in proportion, they will be in proportion by alternation, by inversion, by composition, or by division.*

§ 238. Let $a:b=k:l$, or $\frac{a}{b}=\frac{k}{l}$.

Adding $\pm n$ (§ 42. a),

$$\frac{a}{b}\pm n=\frac{k}{l}\pm n. \quad \therefore \frac{a\pm nb}{b}=\frac{k\pm nl}{l}.$$

$$\therefore \quad a\pm nb:b=k\pm nl:l. \tag{1}$$

Again (§ 235), $\frac{b}{a}=\frac{l}{k}$; and $\frac{b}{a}\pm m=\frac{l}{k}\pm m$;

or $$\frac{b\pm ma}{a}=\frac{l\pm mk}{k}.$$

$$\therefore \quad b\pm ma:a=l\pm mk:k. \tag{2}$$

We have also (§ 234) $a:k=b:l$;

and from (1), $a\pm nb:k\pm nl=b:l=a:k$;

and from (2), $b\pm ma:l\pm mk=a:k=b:l$.

$$\therefore (\S\,231)\ a\pm nb:k\pm nl=b\pm ma:l\pm mk. \tag{3}$$

Now (§ 230. c) ma and mk have the same ratio as a and k; also nb and nl, the same as b and l. Hence,

If either both antecedents or both consequents be increased or diminished by quantities having the same ratio as either consequents or antecedents, the results will be in proportion with either the antecedents or consequents, or with each other.

Thus, if $2:4=6:12$; then $2\pm3:4=6\pm9:12$;

and $2:4\pm1=6:12\pm3$; $2\pm3:4\pm1=6\pm9:12\pm3$.

NOTES. (1.) ma and mk are called EQUIMULTIPLES[p] (i. e. products by a *common multiplier*) of a and k. (2.) If m and n be

(p) Lat. æquus, *equal*, and multiplico, *to multiply* (§ 66. Note w.)

each equal to unity, the formulæ (1) and (2) of this section become identical with (1) and (2) of § 236.

§ 239. Let $a : b = k : l.$

Then $\frac{ma}{b} = \frac{mk}{l}$; $\frac{a}{nb} = \frac{k}{nl}$; and $\frac{ma}{nb} = \frac{mk}{nl}$. § 42. *c, d.*

$\therefore$ $ma : b = mk : l$; $a : nb = k : nl$;

and $ma : nb = mk : nl.$ Hence,

Equimultiples of the antecedents and of the consequents of a proportion will be in proportion, either with the original antecedents, or consequents, or with each other.

Thus, if $2 : 4 = 6 : 12$; then $2\times5 : 4\times7 = 6\times5 : 12\times7$.

Or, $2\times5 : 6\times5 = 4\times7 : 12\times7 = 4 : 12 = 2 : 6$.

Note. We may, obviously, multiply both terms of a ratio (§ 230. *c*) or both the antecedents, or consequents (§ 42. *c, d*) of a proportion, by a common multiplier, without destroying the proportionality.

§ 240. Let $a : b = e : f = g : h = k : l.$

Then $ab = ab$; and (§ 232) $af = be$; $ah = bg$; $al = bk$.

$\therefore$ $a(b+f+h+l) = b(a+e+g+k)$.

$\therefore$ (§ 233) $a+e+g+k : b+f+h+l = a : b = e : f$, &c. Hence,

In any number of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any one of the antecedents is to its consequent.

Thus, if $1 : 2 = 3 : 6 = 4 : 8 = 5 : 10$,

then $1+3+4+5 : 2+6+8+10 = 1 : 2$.

§ 241. Let $a : b = k : l.$

Then (§ 52. N.) $\frac{a^n}{b^n} = \frac{k^n}{l^n}$; or $a^n : b^n = k^n : l^n$. Hence,

Like powers of proportional quantities are proportional.

Thus, if $1 : 4 = 64 : 256$,

then $1^3 : 4^3 = 64^3 : 256^3$;

and $\sqrt{1} : \sqrt{4} = \sqrt{4} : \sqrt{256}$.

Note. The ratio of the squares of two quantities was formerly called the *duplicate*; that of the cubes, the *triplicate*; of the square

and cube roots, the *subduplicate* and *subtriplicate*, ratio of the quantities themselves. The ratio of the square roots of the cubes (i. e. of the *three half powers*) is sometimes called the *sesquiplicate* ratio of the quantities.

§ 242. Let $a : b = k : l$; $e : f = g : h$; and $r : s = x : y$.

Then $$\frac{a}{b} \cdot \frac{e}{f} \cdot \frac{r}{s} = \frac{k}{l} \cdot \frac{g}{h} \cdot \frac{x}{y}; \text{ or } \frac{aer}{bfs} = \frac{kgx}{lhy};$$

or $$aer : bfs = kgx : lhy.$$

The same will evidently hold of any number of proportions. Hence,

The products of the corresponding terms of any number of proportions are proportional.

Thus, if $1 : 3 = 6 : 18$, and $10 : 6 = 15 : 9$,

then $1 \times 10 : 3 \times 6 = 6 \times 15 : 18 \times 9$.

NOTES. (1.) When the terms of two ratios are thus multiplied together, the ratios are said to be *compounded*. (2.) If *equal* ratios are compounded, we obtain the ratio of the powers of the quantities (§ 241).

§ 243. The following exhibits, very briefly, most of the principles above demonstrated (§§ 232–242). If the truth of any of these expressions is not self-evident, write the ratios in the form of fractions.

1. $ar : a = br : b$; or $\frac{ar}{a} = \frac{br}{b}$ (§§ 113. 1; 114; 230. a).

2. $abr = abr$. § 232.

3. $ar : br = a : b$. § 234. See 113. 3.

4. $a : ar = b : br$. § 235.

5. $ar \pm a : a = br \pm b : b$; or $a\,(r \pm 1) : a = b\,(r \pm 1) : b$. § 236.

6. $a\,(r+1) : a\,(r-1) = b\,(r+1) : b\,(r-1)$. § 236. Cor.

NOTE. Other principles may be exhibited in like manner.

§ 244. When the first of four quantities is to the second as the fourth is to the third (i. e. as the *reciprocal* (§ 18) *of the third* is to the *reciprocal of the fourth*), they are said to be *inversely* (§ 235) or *reciprocally* proportional.

Thus, if, on a railroad, a freight train runs 15, and a passenger train 30 miles an hour, their times of passing over equal distances on the road will be inversely or reciprocally proportional to their velocities. That is,

Time of 1st : Time of 2d = Vel. of 2d : Vel. of 1st = $\frac{1}{\text{Vel. of 1st}}$: $\frac{1}{\text{Vel. of 2d}}$; or $T : T' = V' : V = \frac{1}{V} : \frac{1}{V'}$.

If, however, they run equal times, as 3 hours, then the *distances* will be *directly* proportional to their velocities.

VARIATION.

§ 245. These relations are sometimes concisely expressed by saying, that one class of quantities, or, still more concisely, that one quantity *varies* directly or inversely as another. This form of expression is denoted by this symbol $\propto$, or $\doteqdot$, placed between the quantities. Thus, $x \propto y$, or $x \doteqdot y$, (read x varies as y).

Thus, in the examples of the last section, the time is said to *vary* (or to *be*) inversely or reciprocally, and the distance directly, as the velocity. Or, $T \propto \frac{1}{V}$; $D \propto V$.

So, the number of men required to accomplish a work in a given time varies directly as the amount of work; if the amount of work be given, the number of men varies inversely as the time allowed.

§ 246. If $x \doteqdot y$, then we shall have, obviously,

$$x : x' = y : y'; \text{ or } x : y = x' : y'.$$

$$\therefore \qquad \frac{x}{y} = \frac{x'}{y'} = m, \text{ a constant number.} \qquad (1)$$

Also, $$x = my; \text{ and } y = \frac{1}{m}x. \qquad (2)$$

Hence,

When one quantity varies *directly* as another, (1.) the *ratio* of the numbers by which they are expressed is *constant;* and (2.) each is equal to the other *multiplied by* some *constant* number.

§ 247. Let $x \propto \frac{1}{y}$. Then $x : x' = \frac{1}{y} : \frac{1}{y'} = y' : y$.

$\therefore$ $$xy = x'y' = m, \text{ a constant number.} \qquad (1)$$

Also, $$x = \frac{m}{y}; \text{ and } y = \frac{m}{x}. \qquad (2)$$

Hence,

If one quantity varies *reciprocally* as another, (1.) the *product* of the numbers by which they are expressed is *constant;* and (2.) each is equal to a *constant quantity divided by the other.*

a.) The converse of the principles in this and the last section is evidently true.

Hence, (3.) any equation, containing variable quantities, way be written as an expression of variation; and may be simplified by dropping any constant factor on either side.

Also, (4) if all the factors on one side be constant the other side is constant (§§ 246. 1; 247. 1).

Thus, if we have the area of a circle $= \pi R^2$, π being constant[q], then the area varies as the square of the radius; or $$area \propto R^2.$$

So, S representing the space fallen through by a falling body, and T, the time of its descent, if $S = mT^2$, m being constant, then the space varies as the square of the time; or $$S \propto T^2.$$

Again, if the area (A^2) of a rectangle = its base (x) $\times$ its altitude (y); i. e. if $A^2 = xy$, then

the base $= \dfrac{\text{the area}}{\text{the altitude}}$; or $x = \dfrac{A^2}{y} = A^2\dfrac{1}{y}$; and $x \propto \dfrac{1}{y}$; or the base varies inversely as the altitude.

b.) In the last example, the area varies as the product of the base and altitude. So the solidity of a parallelopipedon varies as the product of its length, breadth and thickness.

(q) π, Greek letter *pi*, Eng. *p;* the initial (§ 1. *d*) of περιφέρεια, *periphery, circumference.* In common use, $\pi = 3.14159$ &c. the circumference of the circle whose diameter is unity.

c.) If $x \propto \frac{y}{z}$, then x varies directly as y, and inversely as z. Thus, the weight (W) of a body above the surface of the earth varies directly as its mass (M), and inversely as the square of its distance (D) from the centre of the earth.

That is, $$W \propto \frac{M}{D^2}.$$

§ 248. 1. If, above the surface of the earth, the weight of a given body (i. e. its gravitation towards the earth) varies inversely as the square of its distance from the centre of the earth, how high must the body be raised, that its weight may be only half what it was at the surface?

Let $x =$ the height above the surface of the earth;

$r =$ the radius of the earth; and

$w =$ the weight of the body at the surface.

Then $w : \frac{1}{2}w = \frac{1}{r^2} : \frac{1}{(r+x)^2} = (r+x)^2 : r^2$;

or $$1 : \tfrac{1}{2} = (r+x)^2 : r^2.$$

$\therefore$ $$\tfrac{1}{2}(r+x)^2 = r^2; \text{ or } x^2+2rx = r^2.$$

$\therefore$ $$x = -r \pm r\sqrt{2}.$$

NOTE. Taking the *upper* sign, and finding $\sqrt{2}$ approximately, we have $x = \frac{414}{1000}r$. The *lower* sign gives the distance measured downward (§ 5) through the centre.

2. How far must the body be removed from the surface, that its weight may be w'?

Here we have $$w : w' = (r+x)^2 : r^2;$$

or $$\sqrt{w} : \sqrt{w'} = r+x : r$$

$\therefore$ $$x = -r \pm r\sqrt{\frac{w}{w'}}.$$

3. How much weight will the body lose, if it be removed a given distance (D) from the surface? and what will be its weight there?

Here $$w : w' = (r+D)^2 : r^2$$

$\therefore$ $$w : w-w' = (r+D)^2 : (r+D)^2-r^2. \qquad \text{§ 236.}$$

$$\therefore \qquad w-w'=\frac{(2rD+D^2)w}{r^2+2rD+D^2}, \text{ the loss of weight;}$$

$$w'=\frac{r^2}{(r+D)^2}w.$$

If D is very small compared with r, D^2 may be neglected, and we shall have

$$w-w'=\frac{2D}{r+2D}w.$$

Let $D=1, 2, 5, 10, 100, 1000$ miles, $w=1$ pound, and $r=4000$ miles; and find the values of w' and $w-w'$.

CHAPTER IX.

EQUIDIFFERENT, EQUIMULTIPLE AND HARMONIC SERIES.

I. EQUIDIFFERENT SERIES.

§ 249. A series of quantities such that each *differs* from the preceding by a constant quantity, is called an EQUIDIFFERENT series; and sometimes an *arithmetical* series or *progression.*

a.) Such a series can, of course, be continued to any extent; and its character is determined, if we know any one of its terms and their *common difference.*

Thus, if 7 be one of the terms, and 3 the common difference, we shall have the series,

$$\ldots \quad -5, -2, 1, 4, 7, 10, 13, 16, \quad \ldots$$

Or, if 8 be one of the terms, and -2 the difference, we shall have

$$\ldots \quad 12, 10, 8, 6, 4, 2, 0, -2, -4, \quad \ldots$$

b.) If the common difference be *positive*, the series is called *increasing*; if *negative*, *decreasing*. The first of the series in *a* above, is an increasing series; the second, a decreasing series.

c.) Though every series may be continued without limit, we ordinarily have occasion to consider only some definite number of terms, of which the two extremes are called the *first* and *last* terms.

§ 250. If a be the *first*, and l the *last* of n terms of an equidifferent series, and D their common *difference*, we shall have

1st,	2d,	3d,		$(n-1)$th,	nth,
a,	$a+D$,	$a+2D$,	. .	$a+(n-2)D$,	$a+(n-1)D$ or l;

whence, obviously, $l=a+(n-1)D$. (1)

That is,

The last term is equal to the first term, plus the product of the common difference by the number of terms less one.

NOTE. Of course, the common difference must be taken positive or negative, according as the series is increasing or decreasing.

1. What is the 7th term of the series 1, 3, 5, &c.?

Here $a=1$, $D=2$, and $n=7$.

$\therefore$ $l=a+(n-1)D=1+6\times 2=13$

2. Given $a=25$, $D=-2$, and $n=14$; to find l.

Ans. -1.

3. Given $a=0$, $D=1$, and $n=100$; to find l.

Ans. 99.

§ 251. If s represent the *sum* of n terms of a series, we shall have

$$s=a+(a+D)+(a+2D)\ \ldots +\{\,[a+(n-1)D](=l)\,\};$$

and, writing the terms in the reverse order, obviously

$$s=l+(l-D)+(l-2D)\ \ldots +\{\,[l-(n-1)D)](=a)\,\}.$$

$\therefore$ Adding the equations,

$$2s=(a+l)+(a+l)+(a+l)\ \ldots +(a+l)=n(a+l).$$

$$\therefore \qquad s=\frac{n(a+l)}{2}. \qquad (2)$$ That is,

The sum of any number of terms of an equidifferent series is equal to *the number of terms into half the sum of the extremes.*

1. What is the sum of 20 terms of the series 1, 3, 5, 7, &c.?

Here $a=1$, $D=2$, and $n=20$.

$$\therefore \qquad l=a+(n-1)D=1+19\times 2=39.$$

$$\therefore \qquad s=\tfrac{1}{2}n(a+l)=\tfrac{1}{2}.20(1+39)=400.$$

2. Given $a=1$, $D=1$, and $n=10$; to find l and s.
Ans. $l=10$, $s=55$.

3. Given $a=20$, $D=-2$, and $n=21$; to find l and s.
Ans. $l=-20$, $s=0$.

4. Let $a=20$, $D=-2$, and $n=11$; and find l and s.

§ 252. *a.*) It is obvious from the addition of the two series above (§ 251), that the *sum of any two terms equidistant from the extremes* is equal to *the sum of the extremes.*

Or, beginning with a, the mth term $=a+(m-1)D$;
and, beginning with l, the mth term $=l-(m-1)D$.

Now the sum of these two terms, equidistant from the extremes is $a+l$.

b.) Hence, if the number of terms be *odd*, the *middle* term is *half the sum of the extremes.*

c.) Such a term is called an *equidifferent mean,* and sometimes an *arithmetical* mean.

d.) *The equidifferent mean between two quantities* is found by taking *half their sum.* Thus, the equidifferent mean between 1 and 2 is $1\frac{1}{2}$, or 1.5; between 1 and 1.5, 1.25; between 5 and 15, 10.

e.) The *middle* term is also equal to *the sum of all the terms divided by their number.* For

$$s=\tfrac{1}{2}(a+l)n; \ \therefore \frac{s}{n}=\tfrac{1}{2}(a+l)=\text{the middle term.}$$

NOTE. A *mean* of several quantities, whether they be equidifferent or not, is found by dividing the *sum* of the quantities by their *number*. The mean or average *temperature* for a week or month is found in this way from the several daily temperatures observed during the given period.

§ 253. *f.*) If (§§ 250, 251) we substitute in (2) the value of l in (1), we shall have s in terms of a, D and n. Thus,

$$s = n\tfrac{1}{2}(a+l) = na + \tfrac{1}{2}n(n-1)D = n[a + \tfrac{1}{2}(n-1)D].$$

§ 254. The formulæ, $l = a+(n-1)D$, and $s = \frac{1}{2}n(a+l)$, should be carefully remembered. They contain, it will be observed, five quantities. If any three of these be given, we shall have two equations containing two unknown quantities which may therefore be determined (§§ 124–128).

a.) In fact, from the first, $a = l-(n-1)D$; (3)

$$D = \frac{l-a}{n-1}; \qquad (4)$$

and

$$n = \frac{l-a}{D} + 1. \qquad (5)$$

b.) In like manner, from the second,

$$n = \frac{2s}{a+l}; \qquad (6)$$

$$a = \frac{2s}{n} - l; \qquad (7)$$

$$l = \frac{2s}{n} - a. \qquad (8)$$

§ 255. *c.*) From formula (4) we can *interpolate*[r] any number of equidifferent means between two given extremes. For let it be required to interpolate m intermediate terms between a and b. We shall have the whole number of terms, n, equal to $m+2$. Hence, $n-1 = m+1$, and

$$D\left(=\frac{l-a}{n-1}\right) = \frac{b-a}{m+1}.$$

Hence we have the series,

(*r*) Lat. interpolo, Fr. interpoler, *to insert.*

$$a,\ a+\frac{b-a}{m+1},\ a+2\frac{b-a}{m+1},\ \ .\ \ .\ \ \ a+(m+1)\frac{b-a}{m+1}(=b).$$

1. Interpolate 8 equidifferent means between 1 and 10.

2. Find 6 equidifferent means between 1 and 15.

§ 256. 1. Given $a=1$, $D=1$, and $n=100$; to find l and s. *Ans.* $l=100$, $s=5050$.

2. What is the nth term of the series of example 1 (i. e. the nth term of the natural series 1, 2, 3, 4, &c.)?

Ans. n.

3. What is the sum of n terms of the series 1, 2, 3, &c.?

Ans. $\frac{n(n+1)}{2}$.

4. What is the nth term of the series 1, 3, 5, 7, &c.?

Ans. $2n-1$.

Substitute for n, 1, 2, 3, 4, 5, &c.

5. What is the sum of n terms of the above series of odd numbers, 1, 3, 5, &c.? *Ans.* n^2.

Substitute for n as above.

6. Suppose a body, falling freely to the earth, descends m feet the first second, $3m$ the second second, $5m$ the third, &c. Now if its fall occupy T seconds, how far will it fall in the last second? *Ans.* $(2T-1)m$.

7. How far will it fall in the whole T seconds? i. e. what is the sum of the series, m, $3m$, $5m$, &c., to T terms?

Ans. mT^2.

Substitute, in these two examples, for T, 5, 6, 7, 8, 10, &c. Also find the value of the expressions thus obtained, on the supposition that $m=16\frac{1}{12}$.

II. EQUIMULTIPLE SERIES.

§ 257. A series, such that each term is formed by *multiplying* the term immediately preceding by a *constant multiplier*, is called an EQUIMULTIPLE series; sometimes also a *geometrical* series or *progression*, or a progression *by quotient*.

NOTE. The constant multiplier has been sometimes called the *ratio*. For convenience and distinctness, however, we shall call it the *common multiplier*, or simply the *multiplier*.

a.) Such a series can, of course, be continued to any extent; and its character is determined, if we know any one of its terms and the common *multiplier*.

Thus, if 7 be one of the terms, and 3 the common multiplier, we shall have the series,

$$\ldots\ldots \tfrac{7}{27},\ \tfrac{7}{9},\ 2\tfrac{1}{3},\ 7,\ 21,\ 63,\ \ldots\ldots$$

So, if 8 be one of the terms, and $\frac{1}{2}$ the multiplier, we shall have

$$\ldots\ 32,\ 16,\ 8,\ 4,\ 2,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \ldots$$

b.) If the common multiplier be *greater than unity*, we shall have an *increasing* series; if *less*, a *decreasing* series. The first of the two series in *a*, above, is an increasing, the second a decreasing series.

c.) Though every series may be continued without limit, we ordinarily have occasion to consider only some definite number of terms, of which the two extremes are called the *first* and *last* terms.

§ 258. If a be the *first*, and l the *last* of n terms of an equimultiple series, and m the *common multiplier*, we shall have

$$\begin{array}{llllllll} \text{1st,} & \text{2d,} & \text{3d,} & \text{4th,} & \text{5th,} & & (n-1)\text{th,} & n\text{th,} \\ a, & am, & am^2, & am^3, & am^4, & \ldots & am^{n-2}, & am^{n-1} \text{ or } l. \end{array}$$

Whence, obviously, $\quad l = am^{n-1}. \qquad (1)$

That is, to find the nth term of an equimultiple series,

Multiply the first term by the $(n-1)$*th power of the common multiplier.*

1. What is the 6th term of the series 1, 2, 4, &c.?

Here $a = 1$, $m = 2$, and $n = 6$.

$$\therefore \qquad l(= am^{n-1}) = 1 \times 2^5 = 32.$$

2. Given $a = 3$, $m = 2$, and $n = 10$; to find l.

Ans. 1536.

3. Given $a=64$, $m=\frac{1}{2}$, and $n=8$; to find l.

Ans. $\frac{1}{2}$.

4. Given $a=\$100$, $m=1.06$, and $n=10$; to find l; i. e. to what will \$100 amount in 10 years at 6 per cent. compound interest? *Ans.* $l=\$179.09$.

5. What is the amount (A) of p dollars, at compound interest for t years, at the rate r?

Here we have

$1+r=$ the amount of *one* dollar for one year.

$\therefore\quad p(1+r)=$ " p dollars "

$p(1+r)(1+r)=$ " $p(1+r)$ " "

&c.

Or, $p=$ the amount at the *beginning* of the *first* year;

$p(1+r)=$ " " *second* "

$p(1+r)^2=$ " " *third* "

: : : : :

$p(1+r)^{n-1}=$ " " nth "

$p(1+r)^t=$ " " $(t+1)$th "

i. e. at the *end* of t years.

The successive amounts constitute, obviously, an equimultiple series; in which we have given $a=p$, $m=1+r$, and $n=t+1$; to find $l=A$. *Ans.* $A=p(1+r)^t$.

6. What is the amount of \$50 at 6 per cent. compound interest for 12 years? *Ans.* \$100.61.

7. What sum, at the rate r, will amount to A dollars in t years?

Ans. $p=\dfrac{A}{(1+r)^t}$. See 4, above.

8. What principal at 6 per cent will amount to \$1000, in 10 years? *Ans.* \$558.37.

9. At what rate of compound interest will p dollars amount to A dollars in t years? *Ans.* $r=\left(\dfrac{A}{p}\right)^{\frac{1}{t}}-1$.

Let $p=100$, $A=150$, and $t=8$; &c.

a.) If we have $m<1$, and $n=\infty$, then putting $m=\dfrac{1}{m'}$

(m' being of course >1), we shall find $l\,(=am^{\infty-1})= a\left(\frac{1}{m'}\right)^{\infty-1}=\frac{a}{m'^{\infty-1}}=\frac{am'}{m'^{\infty}}=\frac{am'}{\infty}=0$ (§ 138. 3)[s]. That is,

The last term of a *decreasing infinite* equimultiple series is *zero.*

NOTES. (1.) Of an *infinite* series, there *can be no last* term. And, on the other hand, in forming the terms by multiplication, we can *never* arrive at *zero;* though we must evidently approximate to it. Hence, the inconsistency in speaking of the *last* term of an *infinite* series is compensated by placing it beyond any *finite* number of terms; i. e. at an infinite distance.

(2.) In the two series,

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \&c.;\ \text{and}\ 2,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \&c.,$$

any term whatever of the first is half the corresponding term of the second. Hence, the *last* terms are said to be in the same ratio. Now this comparison can, obviously, be made only between terms at some definite distance from the beginning. That distance, however, can be taken as great as we please; and the terms, consequently, can be brought as near zero (and, therefore, as near *equality*) as we please, while the ratio remains constant. Thus *infinitesimals,* though regarded as equal to zero, may, like finite quantities, have any *definite ratio to each other.*

§ 259. *b.*) It is evident, from the formation of the several terms of an equimultiple series, that the product of *any two terms equidistant from the extremes* must be equal to the *product of the extremes.*

In fact, if a be the first of n terms, the term which has p terms before it will be am^p; the one which has p terms after it, being the $(n-p)$th, will be equal to am^{n-p-1}.

Hence their product

$$am^p\times am^{n-p-1}=a^2m^{n-1}=a\times am^{n-1}=al\ (\S\ 258).$$

c.) Or again, if a be the first term, and m the multiplier, we shall have the $(p+1)$th term $=am^p$.

(s) It is evident that, since $m'>1$, a finite number of factors, each equal to m', may be taken sufficient to produce any finite number whatever. Hence, if we combine an *infinite* number of these factors, *the result* will be *infinite.*

But, if we begin with l, the multiplier is, obviously, $\frac{1}{m}$;

and the $(p+1)$th term $= l\left(\frac{1}{m}\right)^p = \frac{l}{m^p}$.

$$\therefore \qquad am^p \times l\frac{1}{m^p} = al.$$

d.) Hence, if the number of terms be odd, *the product of the extremes will be equal to the square of the middle term* (it being equally distant from the two extremes).

e.) Such a term may be called an *equimultiple mean.* It is sometimes called a *geometrical* mean, and is simply a mean proportional between the extremes (§§ 231. *b*; 232. *b*).

§ 260. If s represent the *sum* of n terms of an equimultiple series, we shall have

$$s = a + am + am^2 + am^3 + \quad . \quad . \quad + am^{n-1}.$$

Multiplying by m,

$$ms = am + am^2 + am^3 + am^4 + \quad . \quad + am^n.$$

Subtracting the first of these equations from the second,

$$ms - s = am^n - a; \text{ or } (m-1)s = a(m^n - 1).$$

$$\therefore \qquad s = \frac{a(m^n-1)}{m-1}. \qquad (2)$$ That is,

To find the *sum* of n terms of an equimultiple series,

Raise the multiplier to the nth power, and subtract 1 *; multiply the remainder by the first term, and divide the product by the multiplier diminished by unity.*

§ 261. *a.*) We have $s = \frac{am^n - a}{m-1}$, § 260.

and $l = am^{n-1}$. § 258.

$$\therefore \qquad s = \frac{lm - a}{m-1}. \qquad (3)$$ That is,

To find the *sum* of n terms of an equimultiple series,

Multiply the last term by the multiplier, subtract the first term, and divide by the multiplier less one.

b.) If $m<1$, both m^n-1 and $m-1$ will be *negative.* In that case it is convenient to change the signs and the order of the terms, thus;

$$s=\frac{a(1-m^n)}{1-m}, \text{ or } s=\frac{a-lm}{1-m}.$$

1. Find the sum of 20 terms of the series 1, 2, 4, 8, &c.

Here $a=1$, $m=2$, and $n=20$;

$$\therefore \quad s=\frac{a(m^n-1)}{m-1}=\frac{1(2^{20}-1)}{2-1}=1{,}048{,}575.$$

2. Given $a=243$, $m=\frac{1}{3}$, and $n=7$; to find l and s.
Ans. $l=\frac{1}{3}$, $s=364\frac{1}{3}$.

3. Given $a=1$, $m=4$, and $n=5$; to find l and s.
Ans. $l=256$; $s=341$.

4. Given $a=1$, $m=\frac{1}{3}$, and $n=6$, to find l and s.
Ans. $l=\frac{1}{243}$; $s=1\frac{121}{243}$.

c.) If $m<1$, and $n=\infty$, we should have, reasoning as in § 258. *a*, $am^n=am^\infty=0$.

$$\therefore \quad s=\frac{a}{1-m}. \tag{4}$$

That is,

The sum of a decreasing infinite equimultiple series is equal to the first term divided by the difference between unity, and the common multiplier.

We might obtain the same result by substituting the value of l (§ 258. *a*) in formula (3) of § 261.

NOTES. (1.) If n is infinite, $n-1$ is infinite also. For, if $n-1$ were finite, n being greater by unity than a finite number, must be finite also. In like manner, *if any finite quantity whatever be subtracted from infinity, the remainder is still infinite.* (2.) Hence, we have $\infty=\infty\pm a$. That is, an *infinite* quantity is not affected by the *addition or subtraction of a finite* quantity.

1. Given $a=1$, $m=\frac{1}{2}$, and $n=\infty$; to find the sum of the series. *Ans.* $s=2$.

2. What is the sum of the infinite series, whose first term is 1, and multiplier $\frac{1}{10}$? *Ans.* $1\frac{1}{9}$.

3. Given a and s in a decreasing infinite series, to find m.

Ans. $m = \frac{s-a}{s}$, i. e. $1-\frac{a}{s}$.

4. Given $a=1$, $s=3$, and $n=\infty$, to find m.

Ans. $m=\frac{2}{3}$.

5. Given m and s, when $n=\infty$, to find a.

Ans. $a=(1-m)s$.

6. Given $m=\frac{1}{5}$, $s=10$, and $n=\infty$, to find a.

Ans. $a=8$.

§ 262. *d.*) Suppose that at the end of one year from the present time, and also at the end of each succeeding year, a man invests a dollars at r per cent. compound interest. What will be the whole amount of his investment and interest at the end of t years?

We shall have the

amount of the *first* investment for			$t-1$ years		$=a(1+r)^{t-1}$.
"	*second*	"	$t-2$	"	$=a(1+r)^{t-2}$;
:	:	:	:	:	:
	last but one	"	1 year		$=a(1+r)$;
	last	"	0	"	$=a$.

Hence, if $A'=$ the *whole* amount, we shall have

$$A'=a[(1+r)^{t-1}+(1+r)^{t-2}\ .\ .\ +(1+r)^2+(1+r)+1)];$$

or (§ 260)
$$A'=a\frac{(1+r)^t-1}{r}. \qquad (1)$$

Note. This is the amount of an *annuity*[t] of a dollars, which has been *forborne* (i. e. left unpaid) t years.

1. Given $a=\$100$, $r=.06$, and $t=10$ years; to find A'.

Ans. $A'=\$1318.08$.

2. Given $a=\$200$, $r=.05$, and $t=8$ years, to find A'.

Ans. $A'=\$1909.82$.

e.) The *present worth* of an annuity for any number of years is, evidently, the same as the present worth of the amount of the annuity (§ 262. *d*); i. e. it is such a sum, as,

(t) Fr. annuite', *yearly payment;* from Lat. annus, *a year.*

put at interest now, will produce that amount in the given time (§ 258. 7).

If p' = the present worth, we shall have (§ 258. 7; 262. *d*)

$$p' = \frac{A'}{(1+r)^t} = \frac{a}{r}\,\frac{(1+r)^t-1}{(1+r)^t} = \frac{a}{r}\left(1-\frac{1}{(1+r)^t}\right). \qquad (2)$$

1. Given $a = \$100$, $r = .06$, and $t = 10$ years; to find p'. *Ans.* $p' = \$736.01$.

2. Given $a = \$500$, $r = .06$, and $t = 12$ years; to find A' and p'. *Ans.* $A' = \$8434.97$; $p' = \$4191.92$.

f.) If the annuity be a *perpetuity*[u] (i. e. if it last *forever*), we have $t = \infty$, and (§ 258. N. *s*) $\frac{1}{(1+r)^t} = 0$.

$\therefore\ p' = \frac{a}{r}$, the sum, evidently, whose annual interest is a.

g.) These formulæ, as well as those relating to compound interest (§ 258. 5–9), will be the same, whether the interest and annuity be payable at the end of each year, or of each half year, quarter, month, day, hour, or other period; r denoting the interest of \$1 for the *given period*, and t, the *number of the periods*.

h.) Or, if r = the interest of \$1 for a year,

t = the number of years, and

n = the number of periods in a year, we shall have

$\frac{r}{n}$ = the interest of \$1 for the given period; and

nt = the number of periods.

$$\therefore\ (\S\ 258.\ 5) \qquad A = p\left(1+\frac{r}{n}\right)^{nt}. \qquad (3)$$

$$\text{Also } (\S\ 262.\ d) \qquad A' = \frac{na}{r}\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]; \qquad (4)$$

$$\text{and } (\S\ 262.\ e) \qquad p' = \frac{na}{r}\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]. \qquad (5)$$

Given $p = \$100$, $r = .06$, $n = 4$, and $t = 3$ years; to find A. *Ans.* $A = \$119.52$.

(*u*) Lat. perpetuitas, *that which lasts forever*.

i.) The interest may be conceived to be payable at each *moment* as the use of the money is enjoyed. In that case, n becomes infinite, and formula (1) reduces to a peculiar form, which will be considered hereafter.

j.) If, while the interest is payable *annually*, the compound interest for a *part* of a year be required, the value of t in the formula of § 258. 5 becomes *fractional.*

Thus, the compound annual interest for *half* a year is $p(1+r)^{\frac{1}{2}}$; for one *third* of a year, $p(1+r)^{\frac{1}{3}}$; &c.

So, for *two and a half years*, we have $A=p(1+r)^{2\frac{1}{2}}$.

§ 263. *k.*) This last result corresponds to the case in which n becomes *fractional* in the formula, $l=am^{n-1}$ of § 258. Nothing prevents our assigning a fractional value to n either in the equidifferent or equimultiple series.

Thus, in the series 1, 3, 5, 7, &c., if $n=3\frac{1}{2}$, we have $l(=a+(n-1)D=1+2\frac{1}{2}\times 2=6$.

So, in the series 1, 2, 4, 8, &c., if $n=3\frac{1}{2}$, we have

$l(=am^{n-1})=1\times 2^{2\frac{1}{2}}=2^{\frac{5}{2}}=32^{\frac{1}{2}}=5.65685$&c. § 258.

NOTE. This, it will be observed, is equivalent to interpolating a single mean, equidifferent or equimultiple, as the case may be, between the *third* and *fourth* terms of the series (§§ 255; 265).

l.) Again, n may, obviously, become *zero*, or *negative.*

Thus (§ 250), let $a=1$, $D=2$, and $n=0$.

Then $l[=a+(n-1)D]=1+(0-1)2=-1$.

If $n=-3$, then $l=1+(-3-1)2=-7$.

Also (§ 258) let $a=1$, $m=2$, and $n=0$.

Then $l(=am^{n-1})=1\times 2^{0-1}=2^{-1}=\frac{1}{2}$ (§ 17).

If $n=-3$, then $l=1\times 2^{-3-1}=2^{-4}=\frac{1}{16}$.

§ 264. The formulæ, $l=am^{n-1}$ (1), $s=\dfrac{a(m^n-1)}{m-1}$ (2) and $s=\dfrac{lm-a}{m-1}$ (3) should be carefully remembered. They contain, it will be observed, five quantities, from any

three of which the other two may, obviously, be found (§ 254).

a.) Thus, from the first, $a = \frac{l}{m^{n-1}}$. (4)

$$m = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}. \tag{5}$$

b.) To find n, we have $m^{n-1} = \frac{l}{a}$. That is, $n-1$ is the exponent of the power to which m must be raised, to produce $\frac{l}{a}$. An equation, in which the unknown quantity is an exponent, is called an *exponential* equation; and is solved by a peculiar process, which we are not yet prepared to investigate.

§ 265. *c.*) From formula (5) we can interpolate (§ 255. N. r) any number of equimultiple means between two given extremes. For, if it be required to interpolate p terms between a and b, we shall have the whole number of terms, n, equal to $p+2$. Hence, $n-1 = p+1$, $l = b$,

and $$m = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{b}{a}\right)^{\frac{1}{p+1}}.$$

Hence we have the series,

$$a,\ a\left(\frac{b}{a}\right)^{\frac{1}{p+1}},\ a\left(\frac{b}{a}\right)^{\frac{2}{p+1}},\ .\ a\left(\frac{b}{a}\right)^{\frac{p}{p+1}},\ a\left(\frac{b}{a}\right)^{\frac{p+1}{p+1}} (= b).$$

1. Interpolate 2 equimultiple means between 3 and 81. Here $p = 2$, $p+1 = 3$, $a = 3$, and $b = 81$.

$$\therefore \quad m = \left(\frac{b}{a}\right)^{\frac{1}{p+1}} = 27^{\frac{1}{3}} = 3.$$

Hence the series is 3, 9, 27, 81.

§ 266. 1. Given $a = \frac{1}{16}$, $m = 2$, and $n = 10$, to find l and s. *Ans.* $l = 32$, $s = 63\frac{15}{16}$.

2. Given $a = 1$, $m = 1$, and $n = 100$; to find l and s. *Ans.* $l = 1$, $s = 100$.

Note. Here we have

$$s\left(= \frac{a(m^n - 1)}{m-1}\right) = \frac{1(1-1)}{1-1} = \frac{0}{0},$$

apparently indeterminate (§ 109. *c*). But when $m=1$, we have (§ 140)

$$\frac{m^n-1}{m-1}=nm^{n-1}=100\times1^{99}=100.$$

3. Given $a=1$, $m=-x$ (where $x<1$), and $n=\infty$, to find s; or, in other words, to find the sum of the decreasing infinite series, $1-x+x^2-x^3+$&c.

Ans. $s=\dfrac{1}{1+x}$.

If we had $x>1$, we should find the same result, but by a different process.

4. Of four terms of an equimultiple series, the product of the two least is 8, and of the two greatest 128. What are the numbers? *Ans.* 2, 4, 8, 16.

5. What is the vulgar fraction equivalent to the repeating decimal .121212 &c.? *Ans.* $\frac{4}{33}$.

NOTES. (1.) This is the same thing as finding the sum of the series $.12+.0012+$&c. to infinity; where $a=.12$, $m=.01$, and $n=\infty$. (2.) In the same way, the value of any repeating decimal may be found. Thus, we have

$$.1111\ \&c.=.1+.01+001+\&c.=.1\div(1-.1)=.1\div.9=\tfrac{1}{9}$$

6. Find the vulgar fraction equal to .1010&c.; 222&c.; .456456&c.; 74357435&c.

HARMONIC SERIES.

§ 267. 1. *Three* numbers are said to be in HARMONICAL[v] proportion, when the first is to the third, as the difference of the first and second is to the difference of the second and third.

Thus, if $a:c=a-b:b-c$, then a, b, and c are in harmonical proportion. So 2, 3 and 6 are in harmonical proportion, because $2:6=3-2:6-3$.

2. *Four* numbers are said to be in harmonical proportion, when the first is to the fourth, as the difference of the

(v) Gr. ἁρμονία, *joining, harmony.*

first and second is to the difference of the third and fourth.

Thus, 6, 8, 12 and 18 are in harmonical proportion; because

$$6 : 18 = 8-6 : 18-12.$$

§ 268. Let a, b, and c be in harmonical proportion.

Then $$a : c = a-b : b-c.$$

$\therefore$ $$ab-ac = ac-bc;\ (1) \text{ or } (a+c)b = 2ac.$$

$\therefore$ $$b = \frac{2ac}{a+c};\ (2) \text{ and } c = \frac{ab}{2a-b}.\quad (3)$$

§ 269. A *harmonic series* or *progression* is one in which *any three consecutive terms are in harmonic proportion.*

Thus, 6, 3, 2, 1.5, 1.2, 1 form a harmonic series, as will be readily seen by forming proportions as in § 267. 1.

§ 270. Let a, b, c, f, g, h, &c., be consecutive terms of a harmonic series. Then (§ 268. 2)

$$b = \frac{2ac}{a+c},\ c = \frac{2bf}{b+f},\ f = \frac{2cg}{c+g},\ \text{\&c.}$$

Dividing unity by both sides of each equation,

$$\frac{1}{b} = \frac{a+c}{2ac};\ \frac{1}{c} = \frac{b+f}{2bf};\ \frac{1}{f} = \frac{c+g}{2cg},\ \text{\&c.}$$

or $$\frac{1}{b} = \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right);\ \frac{1}{c} = \frac{1}{2}\left(\frac{1}{f} + \frac{1}{b}\right);\ \frac{1}{f} = \frac{1}{2}\left(\frac{1}{g} + \frac{1}{c}\right);\ \text{\&c.}$$

$\therefore\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{f}, \frac{1}{g}$, &c. are terms of an equidifferent series (§ 252. *d*). That is,

The reciprocals of the terms of a *harmonic* series constitute an *equidifferent* series.

a.) This principle may be shown otherwise. Thus,

$$ab-ac = ac-bc. \qquad \text{§ 268. 1.}$$

Dividing by abc, $$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}. \qquad \text{§ 249.}$$

b.) Conversely, it can be readily shown, that the reciprocals of the terms of an equidifferent series constitute a harmonical series.

c.) In order, therefore, to interpolate any number of harmonic means between two quantities, or to continue a harmonic series, of which two terms are given, we have only to interpolate a like number of *equidifferent* means between the *reciprocals* of the given terms; or to extend the equidifferent series, of which those reciprocals form a part; and take the reciprocals of the terms so found.

Thus, to insert two harmonical means between 60 and 15, we must insert two equidifferent means between $\frac{1}{60}$ and $\frac{1}{15}$. This will give the equidifferent series $\frac{1}{60}$, $\frac{2}{60}$, $\frac{3}{60}$, $\frac{4}{60}$. Hence, the harmonic series is 60, 30, 20, 15.

The *succeeding* terms of the equidifferent series will be

$$\tfrac{5}{60},\ \tfrac{6}{60},\ \tfrac{7}{60};$$

and the corresponding terms of the harmonic series,

$$12,\ 10,\ 8\tfrac{4}{7},\ \&c.$$

§ 271. One of the most interesting examples of harmonic series consists of the reciprocals of the natural numbers, 1, 2, 3, 4, 5, 6, &c.;

viz. $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6}$, &c.;

or, reducing the first six terms to a common denominator, and taking the numerators,

$$60,\ 30,\ 20,\ 15,\ 12,\ 10.$$

NOTE. This series may be regarded as the origin of the term *harmonical* or *musical* proportion; the name having been applied to this series on account of the perfect harmony produced by six musical strings of equal thickness and tension, and having their lengths in the ratio of these numbers. For the sharpness of the sound produced by a string, is found to be directly as the number of its vibrations in a given time; and the number of vibrations is inversely as the length of the string. Hence, the longest string sounding the key note, the second string will sound the octave; the third will sound the twelfth, or fifth of the octave; the fourth, the fifteenth or double octave; the fifth, the seventeenth or third of the double octave; the sixth, the nineteenth or fifth of the double octave.

CHAPTER X.

PERMUTATIONS AND COMBINATIONS.

§ 272. *Changes in the order* of things arranged together are called PERMUTATIONS[w]. To determine the possible number of changes of this kind, is the object of the theory of permutations.

§ 273. A single individual, as the letter a, can obviously give rise to no question of the kind. But, if a *second* letter, b, be taken, this can be placed either before or after the first; thus ab or ba. Thus,

the permutations of two letters $= 1 \cdot 2 = 2$.

Let there be a *third* letter, c. This may have *three* places in *each* of the permutations of the two letters; thus,

cab, acb, and abc; cba, bca, and bac.

That is, it may stand before each of the other letters, and after them both. Hence, the number of the permutations of *three* things will be

2×3 (or, for symmetry) $1 \cdot 2 \cdot 3 = 6$.

In like manner, a *fourth* letter might stand in *four* places, in *each* of the six preceding permutations, and would give the number of

permutations of *four* things $= 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

So a *fifth* letter might stand in *five* places in *each* of the

(w) Lat. permuto, *to change*.

24 permutations of four letters; giving five permutations for each change of the four. Hence, the number of

permutations of *five* things $= 1 . 2 . 3 . 4 . 5 = 120$.

Thus, when the nth letter is introduced, there being $n-1$ letters in each of the preceding permutations, the new letter can stand in n places in *each* of those permutations; and we shall have the whole number of

permutations of n things $= 1 . 2 . 3 . 4 . 5 \;.\;.\; (n-1)n$.

1. How many permutations can be made with the six vowels a, e, i, o, u, and y? *Ans.* 720.

2. How many permutations can be made in writing the nine digits? *Ans.* 362880.

NOTE. The expression $[n]$ is sometimes used to denote the product $1 . 2 . 3 \;.\;.\; n$. Thus $[10] = 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10$.

§ 274. We sometimes inquire the number of changes in the position of n things taken, not all at once, but *a part at a time.* The changes thus produced are called *arrangements* or *variations.*

To find the number of such arrangements, we must consider that we may write any one of the n letters, as a, before each of the remaining $n-1$ letters. We shall thus have $n-1$ arrangements of n letters taken two and two, in each of which a stands first. We may, in like manner, have $n-1$ arrangements in which b shall stand first; and so for each of the n letters. Hence, the whole number of arrangements of n things taken *two* at a time will be

$$n(n-1).$$

Again, each of these $n(n-1)$ arrangements of n letters taken two at a time can be placed before each of the remaining $n-2$ letters. Hence, the number of arrangements of n things taken by *threes* will be

$$n(n-1)(n-2).$$

In the same way, placing each of these $n(n-1)(n-2)$ arrangements before each of the remaining $n-3$ letters, the

number of arrangements of n things taken *four* at a time will be $$n(n-1)(n-2)(n-3).$$

And, in general, the number of arrangements of n things taken p at a time will be

$$n(n-1)(n-2) \quad . \quad . \quad (n-p+1)\,; \text{ or } [n, n-p+1],$$

by a notation analogous to that of § 273. N.

a.) We have, obviously,

$$n(n-1)\,.\,.\,(n-p+1) = \frac{1\,.\,2\,.\,.\,(n-p)(n-p+1)\,.\,.\,(n-1)n}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,(n-p)}.$$

That is, the number of arrangements of n individuals taken p at a time is equal to the whole number of permutations of n individuals, divided by the number of permutations of $n-p$ individuals; (i. e. by the number of permutations which can be made with the individuals left out of each arrangement).

1. How many arrangements can be made with the 10 Arabic numerals, taken 2 at a time? *Ans.* 90.

2. How many, if they be taken 3 at a time? *Ans.* 720.

3. How many arrangements can be made from the 72 numbers of a lottery, taking 3 numbers upon each ticket? *Ans.* 357840.

b.) If $p = n$, we shall have simply the *permutations* of n things $= n(n-1) \quad . \quad . \quad . \quad . \quad 2\,.\,1$, as in § 273.

§ 275. *Combinations* are the *groups*, that can be formed of individuals *without reference to the order of arrangement;* in other words, groups such, that *no two of them shall be composed of the same individuals.* Thus *ab* and *ba* form two permutations or arrangements, and but one *combination.*

And, in general, whatever be the number of things, only one combination can be formed by taking them all at

once. For two combinations are not different, unless they differ in, at least, one of the individuals contained in them.

Hence, each *combination* of n things may be subjected to $1.2.3..n$ permutations, without affecting the combination. So, if we combine n things, p at a time, each *combination* admits of $1.2.3.....p$ permutations or arrangements.

Hence we shall have only one combination for every $1.2.3......p$ arrangements. If then we divide the number of *arrangements* by the number of *permutations in each arrangement*, we shall have the number of *combinations.*

That is, if we combine n things, p at a time, we shall have

$$\text{No. combinations} = \frac{\text{No. arrangements of } n \text{ things taken } p \text{ \& } p}{\text{No. permutations of } p \text{ things}}.$$

Now the number of arrangements of n things taken p at a time is (§ 274)

$$n(n-1)\ ....\ (n-p+1); \quad \text{or } [n, n-p+1];$$

and the number of permutations of p things is

$$1.2.3\ ....\ p; \quad \text{or } [p].$$

Therefore the number of combinations of n things taken p at a time is

$$\frac{n(n-1)\ ...\ (n-p+1)}{1.2.3\ ....\ p} = \frac{[n, n-p+1]}{[p]}.$$

a.) If the letters denote algebraic quantities, the number of *combinations* of n letters taken p at a time is, evidently, the same as the number of distinct *products* of the quantities taken p at a time.

b.) If n things be taken p at a time, then (§§ 274. *a*; 275)

$$\text{No. of arrangements} = \frac{n(n-1)..(n-p+1)(n-p)..3.2.1}{1.2.3\ .\ .\ (n-p)};$$

$$\text{No. of combinations} = \frac{n(n-1)..(n-p+1)(n-p)..3.2.1}{1.2.3...p.\ 1.2.3...(n-p)}.$$

Again, if n things be taken $n-p$ at a time, we have

$$\text{No. of arrangements} = \frac{n(n-1)\,..\,(n-p+1)(n-p)\,..\,3.2.1}{1.2.3\,....\,p};$$

$$\text{No. of combinations} = \frac{n(n-1)\,..\,(n-p+1)(n-p)\,..\,3.2.1}{1.2.3\,...\,p.1.2.3\,...\,(n-p)}.$$

Hence, the number of combinations of n individuals, is the same whether they be taken p, or $n-p$, at a time.

Thus, the number of combinations of 10 letters will be the same, whether they be taken 3 and 3, or 7 and 7.

c.) The last principle is evident also from the fact, that, for each combination of p things taken, a combination of $n-p$ things must be left.

1. How many products (§ 275. *a*) can be formed of the 6 quantities, a_1, a_2, a_3, a_4, a_5, a_6,^{x} by taking them 1 by 1, 2 by 2, 3 by 3, 4 by 4, and 5 by 5?

Ans. 6, 15, 20, 15, and 6.

2. How many products of 4 quantities taken 1, 2, 3, and 4, respectively, at a time? *Ans.* 4, 6, 4, and 1.

d.) The number of combinations must, of course, be a whole number. Therefore $\frac{[n, n-p+1]}{[p]}$ is a whole number.

(x) These numbers, $_1$, $_2$, $_3$, &c., are used as accents (§ 1. *e*).

CHAPTER XI.

UNDETERMINED COEFFICIENTS.

§ 276. Let the equation

$$Ax^p+Bx^q+Cx^r+\&c.=A'x^{p'}+B'x^{q'}+C'x^{r'}+\&c. \quad (1)$$

be true for *all values* of x (i. e. whatever value may be assigned to x); A, B, C, &c., and A', B', C', &c., being any quantities whatever not equal to zero or infinity, and each member of the equation being arranged according to the ascending powers of x.

It is required to determine the relation existing between the exponents, and also between the coefficients of x in the corresponding terms, on the two sides of the equation.

Dividing both members of the equation by x^p, we have

$$A+Bx^{q-p}+Cx^{r-p}+\&c.=A'x^{p'-p}+B'x^{q'-p}+C'x^{r'-p}+\&c.$$

Now, as this equation is true for all values of x, it is true when $x=0$. But if $x=0$, the first member reduces to A (the exponents of x in all the terms, but the first being >0, i. e. positive); and the second member evidently, becomes *zero*, if $p'>p$; *infinite*, if $p'<p$.

Hence, if $p'>p$, we shall have

$A=0$, which is contrary to the hypothesis;

and, if $p'<p$, $A=\infty$, also contrary to the hypothesis.

We must, therefore, have $p'=p$; which gives $p'-p=0$; and, if $x=0$,

$$A=A'.$$

Hence, removing equal quantities from both sides of (1),

$$Bx^q + Cx^r + \&c. = B'x^{q'} + C'x^{r'} + \&c. \qquad (2)$$

Dividing by x^q, and making $x=0$, we shall prove $q=q'$, and $B=B'$. And, in like manner, $r=r'$, $C=C'$; &c. Hence,

§ 277. *If an equation between two polynomials, functions of x, be true for all values of x, it must have like powers of x on both sides; and the coefficients of the like powers must be severally equal to each other.*

§ 278. *a.*) Let the equation be given of the form,

$$A+Bx+Cx^2+\&c. = A'+B'x+C'x^2+\&c. \qquad (3)$$

Then, making $x=0$, $\quad A=A'$;

and canceling A and A' in (3), dividing by x, and again making $x=0$, $\quad B=B'$; &c.

§ 279. *b.*) Or, again, transpose all the terms of equation (3) to the first member, and arrange with reference to the powers of x. Thus,

$$A-A'+(B-B')x+(C-C')x^2+\&c. = 0. \qquad (4)$$

Making $x=0$, $\quad A-A'=0$; and $\therefore A=A'$.

Then $\quad (B-B')x+(C-C')x^2+\&c. = 0$

Dividing by x, $\quad B-B'+(C-C')x+\&c. = 0.$

Making $x=0$, $\quad B-B'=0$; and $B=B'$; and so on.

Represent $A-A'$ by M; $B-B'$ by N; $C-C'$ by P; &c. Then if we have, for all values of x,

$$M+Nx+Px^2+\&c. = 0,$$

we shall have also

$$M=0;\ N=0;\ P=0;\ \&c. \quad \text{Hence,}$$

§ 280. *If any polynomial of the form, $M+Nx+Px^2$ +&c., be equal to zero for all values of x, each of the coefficients of the several powers of x, must be separately equal to zero.*

c.) An equation, which is true for all values of a variable, is said to be true *independently* of the variable. Such an equation is an absolute equation (§ 37. *d*).

Thus, the equation,

$$(a+x)^2=a^2+2ax+x^2,$$

is true *independently* of x. On the other hand, the equation,

$$1+x^2=2x-x^3,$$

may be true; but its truth *depends* on the value given to x (§ 38).

§ 281. The above principle (§§ 276–280) is the foundation of the *method of* UNDETERMINED COEFFICIENTS; a method of very great utility in the development of functions and the investigation of principles.

1. Develop $\frac{1}{1+x}$ into a series.

Assume $\frac{1}{1+x}=Ax^{-1}+Bx^0+Cx+Dx^2+\&c.$

Then, if $x=0$, we have $1=\infty$; which is absurd.

Again, assume $\frac{1}{1+x}=Ax+Bx^2+Cx^3+\&c.$

Then, if $x=0$, we have $1=0$; which is absurd.

Assume then $\frac{1}{1+x}=A+Bx+Cx^2+Dx^3+\&c.$

Clearing of fractions and transposing,

$$0=\begin{array}{c|c|c|c|l} A & +A & x+B & x^2+C & x^3+\&c. \\ -1 & +B & +C & +D & \end{array}$$

$\therefore\ A-1=0;\ A+B=0;\ B+C=0;\ C+D=0;$ &c.

$\therefore\quad A=1;\ B=-A=-1;\ C=-B=1;$

$D=-C=-1;$ &c.

Introducing the values of A, B, C, &c., we have

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\&c.$$

NOTE. The results of the several suppositions, in this instance, indicate the method of ascertaining the *form* of the series to be assumed. We may generally determine this, before writing the series,

by making $x=0$ in the function to be developed. The series must be taken, so as to become finite, infinite or zero for $x=0$, according as the function becomes finite, infinite or zero for the same value of x.

2. Develop $(a-x)^{-1}$. See § 87. c. 2.

Assume $\quad \dfrac{1}{a-x}=A+Bx+Cx^2+Dx^3+\&c.$

Then (§ 46) $\quad 1=Aa-A \left| \begin{array}{c} x-B \\ +aB \end{array} \right| \begin{array}{c} x^2-C \\ +aC \end{array} \left| \begin{array}{c} x^3-\&c. \\ +aD \end{array} \right|$

$\therefore\ Aa=1;\ aB-A=0;\ aC-B=0;\ aD-C=0;\ \&c.$

$\therefore\quad A=\dfrac{1}{a};\ B=\dfrac{A}{a}=\dfrac{1}{a^2};\ C=\dfrac{B}{a}=\dfrac{1}{a^3};\ D=\dfrac{1}{a^4};\ \&c.$

$\therefore\ \dfrac{1}{a-x}=\dfrac{1}{a}+\dfrac{1}{a^2}x+\dfrac{1}{a^3}x^2+\&c.=\dfrac{1}{a}\left(1+\dfrac{x}{a}+\dfrac{x^2}{a^2}+\&c.\right)$

Or $\quad (a-x)^{-1}=a^{-1}+a^{-2}x+a^{-3}x^2+a^{-4}x^3+\&c.$
$\quad =a^{-1}(1+a^{-1}x+a^{-2}x^2+\&c.)$

3. Develop $(a-x)^{\frac{1}{2}}$.

Assume $\quad (a-x)^{\frac{1}{2}}=A+Bx+Cx^2+Dx^3+\&c.$

Squaring, $\quad a-x=A^2+2ABx+B^2 \left| \begin{array}{c} x^2+2BC \\ +2AC \end{array} \right| \begin{array}{c} x^3+\&c. \\ +2AD \end{array}$

$\therefore\quad A^2=a;\ 2AB=-1;\ B^2+2AC=0;$
$\quad 2BC+2AD=0;\ \&c.$

$\therefore\quad A=a^{\frac{1}{2}};\ B=-\dfrac{1}{2A}=-\dfrac{1}{2a^{\frac{1}{2}}};\ C=-\dfrac{1}{2\,.\,4a^{\frac{3}{2}}};\ \&c.$

$\therefore\quad (a-x)^{\frac{1}{2}}=a^{\frac{1}{2}}-\dfrac{1}{2}\dfrac{x}{a^{\frac{1}{2}}}-\dfrac{1x^2}{2\,.\,4a^{\frac{3}{2}}}-\dfrac{1x^3}{2\,.\,2\,.\,4a^{\frac{5}{2}}}-\&c.=$

$a^{\frac{1}{2}}\left(1-\dfrac{1}{2}\dfrac{x}{a}-\dfrac{1}{2\,.\,4}\dfrac{x^2}{a^2}-\dfrac{1}{2\,.\,2\,.\,4}\dfrac{x^3}{a^3}-\dfrac{5}{2\,.\,2\,.\,4\,.\,8}\dfrac{x^4}{a^4}-\&c.\right).$

4. Decompose $\dfrac{3x-5}{x^2-6x+8}$ into fractions, whose sum is the given fraction, and whose denominators are the factors of the given denominator.

$x^2-6x+8=(x-4)(x-2).$ § 213. 1.

Therefore assume

$$\frac{3x-5}{x^2-6x+8}=\frac{A}{x-4}+\frac{B}{x-2}.$$

$$\therefore \quad \frac{3x-5}{x^2-6x+8}=\frac{A(x-2)+B(x-4)}{x^2-6x+8}. \qquad \S\,118.$$

$\therefore$ $3x-5=A(x-2)+B(x-4)=(A+B)x-(2A+4B)$.

$\therefore$ $A+B=3,\ 2A+4B=5.$ $\therefore$ $A=\frac{7}{2},\ B=-\frac{1}{2}.$

$$\therefore \quad \frac{3x-5}{x^2-6x+8}=\frac{7}{2(x-4)}-\frac{1}{2(x-2)}. \qquad \text{See } \S\,118.\ 3.$$

Otherwise; as the equation,

$$3x-5=A(x-2)+B(x-4),$$

is true for *all* values of x, it is true, when $x=2$ (i. e. when $x-2=0$).

Introducing this value of x, we have

$$6-5=B(2-4)\,;\ \text{and } \therefore B=-\tfrac{1}{2}.$$

Again, if $x=4$ (or $x-4=0$), we have

$$12-5=A(4-2)\,;\ \text{and } \therefore A=\tfrac{7}{2}, \text{ as before.}$$

5. Decompose $\dfrac{a^3+bx^2}{a^2x-x^3}\left(=\dfrac{a^3+bx^2}{x(a-x)(a+x)}\right)$.

Ans. $\dfrac{a}{x}+\dfrac{a+b}{2(a-x)}-\dfrac{a+b}{2(a+x)}$. See § 118. 2.

6. Develop $(1+x)^{-2}\left(=\dfrac{1}{(1+x)^2}=\dfrac{1}{1+2x+x^2}\right)$ in an infinite series.

Ans. $1-2x+3x^2-4x^3+$&c. Compare § 87. *c*. 5.

7. Decompose $\dfrac{3x^2-1}{x(x+1)(x-1)}$.

Ans. $\dfrac{1}{x}+\dfrac{1}{x+1}+\dfrac{1}{x-1}$.

8. Develop $\dfrac{1-x}{1+x}$ in an infinite series.

Ans. $1-2x+2x^2-2x^3+$&c.

CHAPTER XII.

BINOMIAL THEOREM.

§ 282. Any positive integral power of $x+a$ can be found by multiplying $x+a$ into itself the requisite number of times (§ 164). The proper combination of this process with division and with the extraction of roots will give negative and fractional powers (§ 163).

But this process, when applied even to positive integral powers, beyond a few of the lowest, becomes tedious; its application to negative and fractional powers would be extremely inconvenient.

The BINOMIAL THEOREM enables us to find immediately *any power of a binomial*, whether the exponent be *positive* or *negative*, *integral* or *fractional.*

§ 283. Let it be required to find the nth power of $x+a$.

I. Let n be a *positive integer*.

Then the nth power of $x+a$ is the product of n factors each equal to $x+a$; i. e.

$$(x+a)^n = (x+a)(x+a)(x+a) \ldots \text{ to } n \text{ factors.}$$

To find how the terms of these factors are combined in the terms of the product, multiply together n unequal factors, $x+a_1$, $x+a_2$, $x+a_3$, ... $x+a_n$.

Then $$(x+a_1)(x+a_2) = x^2 + \left.\begin{array}{l} a_1 \\ +a_2 \end{array}\right| x + a_1a_2.$$

$$(x+a_1)(x+a_2)(x+a_3) = x^3 + \left.\begin{array}{l} a_1 \\ +a_2 \\ +a_3 \end{array}\right| x^2 + \left.\begin{array}{l} a_1a_2 \\ +a_1a_3 \\ +a_2a_3 \end{array}\right| x + a_1a_2a_3.$$

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)=$$

$$\begin{array}{r|l|l|l}
x^4+a_1 & x^3+a_1a_2 & x^2+a_1a_2a_3 & x+a_1a_2a_3a_4. \\
+a_2 & +a_1a_3 & +a_1a_2a_4 & \\
+a_3 & +a_1a_4 & +a_1a_3a_4 & \\
+a_4 & +a_2a_3 & +a_2a_3a_4 & \\
 & +a_2a_4 & & \\
 & +a_3a_4 & &
\end{array}$$

Hence we find, that, so far as we have proceeded, (1.) The *exponent* of x in the first term is equal to *the number of factors;* and (2.) *diminishes by unity* in each of the following terms till it becomes zero; also (3.) the *coefficient* of x in the *first* term is *unity;* (4.) in the *second* term it is the *sum of the second terms* of the binomial factors; (5.) in the *third* term, the *sum of the products* of the second terms taken *two and two;* (6.) in the *fourth* term, the sum of the products taken *three and three;* and so on, that in the *last* term being the product of *all* the second terms.

To show the universality of this law, let us suppose that we have found it true for n—1 factors, and see whether it will hold good for n factors (§ 96. N. 1). Thus, suppose

$$(x+a_1)(x+a_2)(x+a_3)\ \ldots\ (x+a_{n-1})=$$

$$\begin{array}{r|r|r|l}
x^{n-1}+a_1 & x^{n-2}+a_1a_2 & x^{n-3}+a_1a_2a_3 & x^{n-4}..+a_1a_2..a_{n-1} \\
+a_2 & +a_1a_3 & +a_1a_2a_4 & \\
: & : & \&c. & \\
: & +a_1a_{n-1} & & \\
+a_{n-1} & : & & \\
 & +a_{n-2}a_{n-1} & &
\end{array}$$

Now introducing the nth factor,

$$(x+a_1)(x+a_2)(x+a_3)\ \ldots\ (x+a_n)=$$

$$\begin{array}{r|r|r|l}
x^n+a_1 & x^{n-1}+a_1a_2 & x^{n-2}+a_1a_2a_3 & x^{n-3}..+a_1a_2..a_n. \\
+a_2 & +a_1a_3 & +a_1a_2a_4 & \\
: & : & : & \qquad\qquad (1) \\
+a_{n-1} & +a_{n-2}a_{n-1} & +a_1a_2a_n & \\
+a_n & +a_1a_n & \&c. & \\
 & : & & \\
 & +a_{n-1}a_n & &
\end{array}$$

Hence (§ 283. 1–6),

§ 284. (1.) The law of the exponents is obviously the same as before.

(2.) The coefficient of x in the *first* term is *unity*, as before.

(3.) The coefficient of x in the second term is the *sum of the second terms* of the n factors.

(4.) The coefficient of x in the third term is the sum of the products of the $n-1$ second terms taken two and two, and also of the products of those $n-1$ terms by the new term a_n; hence it is the sum of the products of the second terms of the n binomials taken *two and two*.

(5.) The coefficient of x in the *fourth* term is composed of the several products of the $n-1$ second terms taken three and three, and also of their products taken two and two multiplied by the new quantity a_n; i. e. it is the sum of all the products of the n second terms taken *three and three*.

(6.) The *last* term is evidently the product of *all* the second terms taken together; or, which is the same thing, the sum of the products of the n second terms taken n and n. For there can be only one such product (§ 275).

Hence, if the above law is true for $n-1$ factors, it is true for n factors. But we have seen, that it is true for 4 factors; it is therefore true for 5, for 6, 7, &c. That is, it is *universal.*

§ 285. Now, if $a_1, a_2, a_3, \ldots . a_n$ are each equal to a, the coefficient of x in the second term will be na; each term in the *third* coefficient of x will be a^2; each term in the *fourth,* a^3; and so on; each term in the nth coefficient of x being a^{n-1}.

Moreover, a^2, in the coefficient of x in the *third* term, will be repeated as many times as there can be products of n quantities taken *two and two;* that is $\frac{n(n-1)}{1 \, . \, 2}$ (§ 275. a).

Also a^3, in the coefficient of x in the *fourth* term, will be repeated as many times, as there can be products of n quantities taken *three and three;* that is $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$.

Hence, we shall have (n being a *positive integer*)

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{1\,.\,2}x^{n-2}a^2 + \&c. \quad .\ .\ .\ .\ + a^n.$$

That is, (1.) *the first term of any positive integral power of a binomial is equal to that power of the first or leading term of the binomial;* (2.) *the exponent of the first term of the binomial diminishes by unity, till it becomes* 0 *; and the exponent of the other term increases by unity from* 0 *to* n.

(3.) *The coefficient of the first term is unity;* *and* (4.) *that of the second term* (i. e. of both x and a) *is* n.

(5.) *The coefficient of any term whatever after the first is found by multiplying the coefficient of the preceding term by the exponent of the leading quantity in that term, and dividing by the number of terms preceding the required term.*

a.) The exponent of the leading quantity becoming 0 in the $(n+1)$th term, the next coefficient found will be 0; and the series will terminate, consisting, as is evident, of $n+1$ terms.

b.) The number of combinations of n things is the same, whether they be taken p and p, or $n-p$ and $n-p$, (§ 275. *b*).

Hence the coefficient of the term, which has p terms before it, is equal to the coefficient of the term, which has $n-p$ terms before it, or p terms after it.

Consequently, if we find the coefficients of *the first half of the terms,* we have also the coefficients of the *last half* in the reverse order.

c.) The last remark is also evidently true, from the fact, that $(a+x)^n = (x+a)^n$, and there is no reason why we should begin with x^n rather than with a^n. Thus,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\,.\,2}a^{n-2}x^2 + \quad .\ .\ .\ \quad + x^n.$$

§ 286. 1. What is the square of $x+a$?

Here we have $n=2$.

$$\therefore \quad (x+a)^n=(x+a)^2=x^n+nx^{n-1}a+\frac{n(n-1)}{1.2}x^{n-2}a^2$$
$$+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3+\&c.=x^2+2xa+\frac{2(2-1)}{1.2}x^0a^2$$
$$=x^2+2xa+a^2.$$

2. $(x+a)^6=$ what?

Here $n=6$.

$$\therefore \quad (x+a)^6=x^6+6x^5a+\frac{6.5}{1.2}x^4a^2+\frac{6.5.4}{1.2.3}x^3a^3+$$
$$\frac{6.5.4.3}{1.2.3.4}x^2a^4+\frac{6.5.4.3.2}{1.2.3.4.5}xa^5+\frac{6.5.4.3.2.1}{1.2.3.4.5.6}a^6$$
$$=x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6.$$

3. $(a+x)^3=$ what? $(x+a)^4$? $(1+x)^4$?

§ 287. *d.*) If, in the general formula (§ 285. *c*), we put $-x$ for $+x$, we have

$$(a-x)^n=a^n-na^{n-1}x+\frac{n(n-1)}{1.2}a^{n-2}x^2-\&c.;$$

the terms containing the *odd* powers of x being *negative.*

1. $(a-x)^5=$ what? $(a-x)^2$? $(a-x)^3$?

2. $(a^2-x^2)^5=$ what?

Ans. $(a^2)^5-5(a^2)^4x^2+10(a^2)^3(x^2)^2-10(a^2)^2(x^2)^3+5(a^2)(x^2)^4-(x^2)^5$; or, reduced,

$$a^{10}-5a^8x^2+10a^6x^4-10a^4x^6+5a^2x^8-x^{10}.$$

3. $(x^2\pm 2ax)^3=$ what?

Ans. $x^6\pm 6ax^5+12a^2x^4\pm 8a^3x^3$.

Make $x^2=b$, and $2ax=c$; develop $(b+c)^3$, and substitute the values of b and c.

e.) The formulæ (§ 285. *c*; 287. *d*) may be put under another form. For

$$a\pm x=a\left(1\pm\frac{x}{a}\right). \quad \therefore (a\pm x)^n=a^n\left(1\pm\frac{x}{a}\right)^n.$$

$$\therefore \quad (a\pm x)^n = a^n\Big(1 \pm n\frac{x}{a} + \frac{n(n-1)}{1\,.\,2}\,\frac{x^2}{a^2} \pm \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,\frac{x^3}{a^3}$$

$$\ldots\ldots + \frac{n(n-1)(n-2)\;.\;.\;(n-2p+1)}{1\,.\,2\,.\,3\;.\;.\;(2p-1)2p}\,\frac{x^{2p}}{a^{2p}} \pm \&c.\Big).$$

NOTE. The above demonstration proceeds upon the supposition that n is a *positive integer;* and is, of course, applicable to that case only.

§ 288. II. *Whatever* be the value of n, whether *integral* or *fractional, positive,* or *negative,* let it be assumed, that

$$(x+y)^n = A + By + Cy^2 + Dy^3 + Ey^4 + \&c.; \qquad (1)$$

A, B, C, &c., being functions (§ 26) of x and n, and entirely independent of y.

NOTE. There can be no negative powers of y, because $(x+y)^n$ is not necessarily infinite when $y=0$. There must, moreover, be a term containing y^0, because $(x+y)^n$ is not necessarily zero, when $y=0$ (§ 281. N.).

§ 289. 1. Let n be a *positive fraction,* $\frac{p}{q}$ (p and q being both integers).

We have $\qquad (x+y)^{\frac{p}{q}} = x^{\frac{p}{q}}\left(1+\frac{y}{x}\right)^{\frac{p}{q}}.$ $\qquad$ § 287. *e.*

Assume $\qquad \left(1+\frac{y}{x}\right)^{\frac{p}{q}} = 1 + P\frac{y}{x} + \&c.,$ $\qquad$ for all values of $\frac{y}{x}$.

Then $\qquad \left(1+\frac{y}{x}\right)^{p} = \left(1+P\frac{y}{x}+\&c.\right)^{q}.$ $\qquad$ § 52. N.

$\therefore$ (§ 285) $\qquad 1 + p\frac{y}{x} + \&c. = 1 + qP\frac{y}{x} + \&c.;$

the remaining terms on both sides containing only higher powers of $\frac{y}{x}$.

$\therefore$ (§ 277) $\qquad p = qP;$ or $P = \frac{p}{q} = n.$

$$\therefore \quad (x+y)^{\frac{p}{q}} = x^{\frac{p}{q}}\left(1 + \frac{p}{q}\,\frac{y}{x} + \&c.\right) = x^{\frac{p}{q}} + \frac{p}{q}x^{\frac{p}{q}-1}y + \&c.$$

Hence, in the *first two* terms, the same law prevails with

the positive *fractional*, as with the positive *integral* exponent (§ 285. 1, 3, 4).

2. Let n be *negative*. Then

$$(x+y)^{-n}=\frac{1}{(x+y)^n}=\frac{1}{x^n+nx^{n-1}y+\&c.}.$$

$$\therefore \qquad (x+y)^{-n}=x^{-n}-nx^{-n-1}y+\&c., \qquad \text{by division;}$$

the remaining terms, evidently, containing successively lower powers of x and higher powers of y.

Hence, again, the *first two* terms follow the same law with the *negative*, as with the *positive* exponent.

Hence, universally, whatever be the value of n (i. e. whether it be *positive* or *negative*, *integral* or *fractional*),

§ 290. The *first two terms* of $(x+y)^n$ are $x^n+nx^{n-1}y$.

We have, therefore, in the series (1) of § 288,

$A=x^n$, and $B=nx^{n-1}$; and the series may be written

$$(x+y)^n=x^n+nx^{n-1}y+Cy^2+Dy^3+Ey^4+\&c.$$

§ 291. Let now each of the quantities x and y be successively increased by any quantity whatever h. The function $(x+y)^n$ will, obviously, undergo an equal change in each case; i. e.

$$(x+\overline{h+y})^n=(\overline{x+h}+y)^n.$$

NOTE. The quantity, h, by which x and y are increased, is called an *increment*(y) of x and y.

1. Adding h to y, series (1) of § 288 becomes

$$[x+(y+h)]^n=A+B(y+h)+C(y+h)^2+D(y+h)^3+\&c.;$$

$$\text{or}\quad [x+(y+h)]^n=A+By+Cy^2+Dy^3+Ey^4+\&c. \qquad (2)$$
$$+Bh+2Cyh+3Dy^2h+4Ey^3h+\&c.;$$
$$\&c. \qquad \&c. \qquad \&c.$$

writing only the terms containing h^0 and h^1.

2. The substitution of $x+h$ for x will, of course, produce no change in y, or in the manner in which it enters into the

(y) Lat. incrementum, *an increase*.

expression; but it will produce a change in each of the coefficients, A, B, C, &c. For, as these coefficients are functions of x, they will, in general, change their value whenever the value of x is changed. Therefore, the powers of y remaining as they are, their coefficients will be what A, B, C, &c. become, when x is changed into $x+h$; i. e. they will be the same functions of $x+h$, as A, B, C, &c. are of x.

Representing, then, by A_{x+h}, B_{x+h}, C_{x+h}, &c., the values assumed by A, B, C, &c., when x becomes $x+h$, we shall have

$$[(x+h)+y]^n = A_{x+h} + B_{x+h}y + C_{x+h}y^2 + D_{x+h}y^3 + \&c. \quad (3)$$

§ 292. Now we have already found

$$A = x^n, \text{ and } B = nx^{n-1}.$$

$$\therefore \quad A_{x+h} = (x+h)^n; \text{ and } B_{x+h} = n(x+h)^{n-1}. \quad (4)$$

But we do not know, what functions C_{x+h}, D_{x+h}, &c., are of $x+h$, because we do not know what functions C, D, &c. are of x. In other words, we do not know what C, D, &c. will be, when $x+h$ is substituted in them for x, because we do not know what they are now.

Let it be assumed, then, that

$$C_{x+h} = C + C'h + \&c.; \quad D_{x+h} = D + D'h + \&c.; \quad (5)$$

and so on; and assume, for symmetry

$$A_{x+h}[=(x+h)^n] = A + A'h + \&c.;$$

$$\text{and} \quad B_{x+h}[=n(x+h)^{n-1}] = B + B'h + \&c.$$

Notes. (1.) This supposition, evidently, involves no absurdity (§ 281. N); for, when $h=0$, the expressions (5) severally reduce to C, D, &c., as they ought, being then simply functions of x as at first (§ 288).

(2.) It will be observed, that A, B, C, &c., in these assumed values, are the primitive *undetermined coefficients*, functions of x (§ 288); and that A', B', C', &c. are the coefficients of h^1 in the several expressions, when $x+h$ is substituted for x.

(3.) If the variable of a function is increased, and the function developed, *the coefficient of the first power of the increment* is a quantity very much employed in analytical investigations; and is called the first *derived function*, or the first *derivative*, or *derivate*, or

differential coefficient, of the primitive function. Thus A' is the first derived function, or derivate of A; B', of B; &c.

(4.) In like manner, if $x+h$ be substituted for x in A', B', &c., the coefficient of h^1 is the first derived function, or derivate of A', B', &c.; and may be called the *second derived function*, or *second derivate* of A, B, C, &c; and may be represented by A'', B'', &c. The same process deduces from the *second* derivate, a *third*, A''', B''', &c.; from the *third*, a *fourth*, A'''', B'''', &c.; and so on.

§ 293. Substituting for A_{x+h}, B_{x+h}, C_{x+h}, &c., the values (5) assumed above (§ 292), and writing as in (2) of § 291 only the terms containing h^0 and h^1, we have

$$[(x+h)+y]^n = A+A'h+\&c.+(B+B'h+\&c.)y+(C+C'h+\&c.)y^2+\&c.;$$

or, arranging according to the powers of h,

$$\begin{aligned}[(x+h)+y]^n = {} & A+By+Cy^2+Dy^3+Ey^4+\&c. \qquad (6)\\ & +(A'+B'y+C'y^2+D'y^3+\&c.)h\\ & +\&c.\end{aligned}$$

Equating (§ 291) the second members of (2) and (6),

$$\left.\begin{aligned}& A+By+Cy^2+\&c.\\ & +(B+2Cy+\&c.)h\\ & +\&c.\end{aligned}\right\} = \left\{\begin{aligned}& A+By+Cy^2+\&c. \qquad (7)\\ & +(A'+B'y+C'y^2+\&c.)h\\ & +\&c.\end{aligned}\right.$$

As this equation is true for all values of h, the coefficients of like powers of h are severally equal (§ 277).

The coefficients of h^0 are *identically* (§ 37. c) the same. Passing then to the coefficients of h^1,

$$B+2Cy+3Dy^2+\&c. = A'+B'y+C'y^2+D'y^3+\&c. \quad (8)$$

Again, as this equation is true for all values of y, the coefficients of like powers of y are severally equal (§ 277).

$$\therefore \quad B=A';\ 2C=B';\ 3D=C';\ 4E=D';\ \&c.$$

$$\text{or}\quad B=A';\ C=\tfrac{1}{2}B';\ D=\tfrac{1}{3}C';\ E=\tfrac{1}{4}D';\ \&c. \quad (9)$$

That is,

a.) B is found by substituting $x+h$ for x in A and taking the coefficient of h^1, viz. A'.

C is found by substituting $x+h$ for x in B (i. e. in A'), and taking half the coefficient of h^1, viz. $\frac{1}{2}B'(=\frac{1}{2}A'')$. See § 292. N. 4.

D is found by substituting $x+h$ for x in C $(=\frac{1}{2}B'=\frac{1}{2}A'')$, and taking one third of the coefficient of h^1, viz. $\frac{1}{3}C'\left(=\frac{B''}{2.3}=\frac{A'''}{2.3}\right)$; and so on. Thus, substituting these values of B, C, &c. in (1) of § 288,

$$(x+y)^n=A+A'y+\frac{A''}{2}y^2+\frac{A'''}{2.3}y^3+\frac{A^{IV}}{2.3.4}y^4+\&c. \quad (10)$$

b.) Or, in other words, B is the first derivate of A; C is half the first derivate of B, i. e. half the *second* derivate of A (§ 292. N. 4); D is one third of the first derivate of C, i. e. one sixth of the second derivate of B, i. e. one sixth of the third derivate of A; and so on.

§ 294. Now we have (§§ 290; 292)

$$A=x^n\text{; and } A_{x+h}=(x+h)^n;$$

or
$$A+A'h+\&c.=x^n+nx^{n-1}h+\&c.$$

$$\therefore \qquad A'=nx^{n-1}. \qquad \text{§ 277.}$$

$$\therefore \qquad B(=A')=nx^{n-1},$$

as we found it before (§ 290).

In like manner, $\qquad B_{x+h}=n(x+h)^{n-1}$;

or
$$B+B'h+\&c.=nx^{n-1}+n(n-1)x^{n-2}h+\&c.$$

$$\therefore \qquad B'=n(n-1)x^{n-2}. \qquad \text{§ 277.}$$

$$\therefore \qquad C(=\tfrac{1}{2}B'=\tfrac{1}{2}A'')=\frac{n(n-1)}{1.2}x^{n-2}.$$

So $\qquad C_{x+h}=\frac{n(n-1)}{1.2}(x+h)^{n-2}$; or

$$C+C'h+\&c.=\frac{n(n-1)}{1.2}x^{n-2}+\frac{n(n-1)(n-2)}{1.2}x^{n-3}h+\&c.$$

$$\therefore \quad D(=\tfrac{1}{3}C'=\frac{1}{2.3}B''=\frac{1}{2.3}A''')=\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}.$$

&c.

$$\therefore \qquad (x+y)^n = x^n + \frac{n}{1}x^{n-1}y + \frac{n(n-1)}{1\,.\,2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}y^3 + \&c. \tag{11}$$

a.) It will not be necessary actually to make the substitution of $x+h$ for x. For the coefficient of each power of y is of the form Mx^n; and the two first terms of $M(x+h)^n$ are $Mx^n + Mnx^{n-1}h$ (§ 290). Hence,

The DERIVATIVE *of each coefficient of y in the series is found by multiplying that coefficient by the exponent of x, and diminishing that exponent by unity.* Therefore,

To find the coefficient of any power of y,

§ 295. *b.*) ***Multiply the coefficient of the preceding term by the exponent of x in that term, diminish the exponent by unity, and divide by the number of terms preceding the required term.***

Thus, the coefficient of y^p (i. e. of the term which has p terms before it) will be $\dfrac{n(n-1)(n-2)\,..\,(n-p+1)}{1\,.\,2\,.\,3\,\,.\,.\,.\,.\,p}x^{n-p}$. (12)

c.) The term $\dfrac{n(n-1)(n-2)\,.\,.\,.\,.\,.\,.\,(n-p+1)}{1\,.\,2\,.\,3\,\,.\,.\,.\,.\,.\,.\,p}x^{n-p}y^p$

is called the *general term* of the series; because if we make in it $p=1, 2, 3,$ &c., successively, we shall have the corresponding terms of the series.

d.) If n is a *positive integer, the series will terminate,* as we have seen (§ 285. *a*). But, if n is *negative,* the subtraction of unity will *numerically increase* the exponent without limit; and, if n is a *positive fraction,* the subtraction of whole units will first render the exponent negative, and then numerically increase it in like manner. Hence, if n be either *negative,* or *fractional, the series will be infinite.*

e.) The sum of the exponents of x and y in each term is, evidently, *constant,* and always equal to n.

f.) If $-y$ be substituted for $+y$, the terms containing the

odd powers of y will, of course, be *negative* (§ 287). Thus,

$$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{1.2}x^{n-2}y^2 - \&c. \quad (13)$$

g.) Let $x=1$, and $y=1$, in formula (11). Then

$$(1+1)^n = 2^n = 1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\&c.$$

Hence, in *any power whatever* of the *sum* of two quantities, *the sum of the coefficients is equal to that same power of* 2.

h.) Let $x=1$, and $y=1$, in formula (13). Then

$$(1-1)^n = 0 = 1-n+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\&c.$$

Hence, in *any power whatever* of the *difference* of two quantities, *the sum of the coefficients is equal to zero;* i. e. the sum of the *positive*, is equal to the sum of the *negative* coefficients.

i.) Let $n=\frac{p}{q}$. Then

$$(x\pm y)^{\frac{p}{q}} = x^{\frac{p}{q}} \pm \frac{p}{q}x^{\frac{p}{q}-1}y + \frac{\frac{p}{q}\left\{\frac{p}{q}-1\right\}}{1.2}x^{\frac{p}{q}-2}y^2 \pm \&c.$$

$$= x^{\frac{p}{q}} \pm \frac{p}{q}x^{\frac{p}{q}-1}y + \frac{p(p-q)}{1.2.q^2}x^{\frac{p}{q}-2}y^2 \pm \&c. \quad (14)$$

$$= x^{\frac{p}{q}}\left\{1 \pm \frac{p}{q}\frac{y}{x} + \frac{p(p-q)}{1.2\ q^2}\frac{y^2}{x^2} \pm \frac{p(p-q)(p-2q)}{1.2.3\ \ q^3}\frac{y^3}{x^3} + \right.$$

$$\left.\frac{p(p-q)(p-2q)(p-3q)}{1.2.3.4\ \ q^4}\frac{y^4}{x^4} \pm \&c.\right\}. \quad (15)$$

§ 296. 1. $(a+x)^{-1} =$ what? See § 87. *c*. 1.

Here $n=-1$.

$$\therefore \quad (a+x)^{-1} = a^{-1} - a^{-2}x + a^{-3}x^2 - a^{-4}x^3 + \&c.$$

$$= a^{-1}(1 - a^{-1}x + a^{-2}x^2 - \&c.)$$

$$= \frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2} - \&c.\right).$$

2. $(a-x)^{-1}=$ what? See § 87. *c.* 2.

3. $(1+u^2)^{-1}=$ what? See § 87. *c.* 4.

4. $(a+x)^{-2}=$ what? See § 87. *c.* 5.

5. $(a+x)^{\frac{1}{2}}=$ what? Compare §§ 173. 1; 281. 3.

Ans. $a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}x+\frac{\frac{1}{2}(-\frac{1}{2})}{1\,.\,2}a^{-\frac{3}{2}}x^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2\,.\,3}a^{-\frac{5}{2}}x^3+$&c.

$=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1}{2.4}a^{-\frac{3}{2}}x^2+\frac{1.3}{2.4.6}a^{-\frac{5}{2}}x^3-\frac{1.3.5}{2.4.6.8}a^{-\frac{7}{2}}x^4+$

&c. $=a^{\frac{1}{2}}\left(1+\frac{1}{2}\frac{x}{a}-\frac{1}{8}\frac{x^2}{a^2}+\frac{1}{16}\frac{x^3}{a^3}-\frac{5}{128}\frac{x^4}{a^4}+\text{\&c.}\right).$

This might have been put under the form

$$a^{\frac{1}{2}}\left(1+\frac{x}{a}\right)^{\frac{1}{2}},$$

and the last form of the answer would have been obtained immediately. The following examples may be similarly reduced. It is well to solve them under both forms.

Also let $a=100$, and $x=1$; $a=400$, and $x=8$; &c.

6. $(a^2+x^2)^{\frac{1}{2}}=$ what? $(R^2-x^2)^{\frac{1}{2}}$? See § 173. 2.

7. $\sqrt[4]{(a^2-x^2)^3}\left[=(a^2-x^2)^{\frac{3}{4}}=a^{\frac{3}{2}}\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{4}}\right]=$ what?

Ans. $a^{\frac{3}{2}}\left(1-\frac{3}{2^2}\frac{x^2}{a^2}-\frac{3}{2^5}\frac{x^4}{a^4}-\frac{5}{2^7}\frac{x^6}{a^6}-\frac{45}{2^{11}}\frac{x^8}{a^8}-\text{\&c.}\right).$

§ 297. 1. Extract the cube root of 65.

$$65=64(1+\tfrac{1}{64}).$$

$\therefore (65)^{\frac{1}{3}}=(64)^{\frac{1}{3}}(1+\frac{1}{64})^{\frac{1}{3}}=4(1+\frac{1}{64})^{\frac{1}{3}}$

$=4(1+\frac{1}{3}\cdot\frac{1}{64}+\frac{\frac{1}{3}\,.\,-\frac{2}{3}}{1\,.\,2}\left(\frac{1}{64}\right)^2+\frac{\frac{1}{3}\,.\,-\frac{2}{3}\,.\,-\frac{5}{3}}{1\,.\,2\,.\,3}\left(\frac{1}{64}\right)^3+$&c.)

$=4\left(1+\frac{1}{3\,.\,2^6}-\frac{1}{3\,.\,6\,.\,2^{11}}+\frac{5}{3\,.\,6\,.\,9\,.\,2^{17}}-\text{\&c.}\right)$

$=4(1+\frac{1}{192}-\frac{1}{36{,}864}+$&c.$)=4.020{,}724$&c.

2. What is the tenth root of $1056(=1024+32)$?

$$(1056)^{\frac{1}{10}}=1024^{\frac{1}{10}}(1+\tfrac{32}{1024})^{\frac{1}{10}}=2(1+\tfrac{1}{32})^{\frac{1}{10}}$$

$$=2\left(1+\frac{1}{10}\cdot\frac{1}{32}-\frac{9}{200}\cdot\left(\frac{1}{32}\right)^2+\frac{171}{6000}\left(\frac{1}{32}\right)^3-\&c.\right)$$

$$=2\left(1+\frac{1}{320}-\frac{9}{204,800}+\frac{171}{196,608,000}-\&c.\right)=2.006,164.$$

§ 298. Let $a-b+c-f+g-h+k-l+$&c. be a *converging* series consisting of *terms alternately positive and negative.* It is required to determine the *degree of approximation* attained when we stop at a particular term.

If we stop at a negative term, as f, there will remain a set of *positive* quantities, $g-h$, $k-l$, &c., to be added to obtain the true sum of the series. Hence, if we stop at a *negative* term, the sum of the terms taken is *too small.*

If, on the other hand, we stop at a positive term, as g, there will remain a set of *negative* quantities, $-h+k$, $-l+m$, &c., to be added to obtain the true sum of the series. Hence, if we stop at a *positive* term, the sum of the terms taken is too *great.*

Now the sum of the terms, before g was added, being too small, and, after g was added, too great, the error in the first instance must have been less than g.

In the same manner, it is evident, that, if we stop at g, the error is numerically less than h.

Hence, whatever number of terms of a converging series whose terms are alternately positive and negative we take, *the error will be numerically less than the next succeeding term.*

CHAPTER XIII.

DIFFERENCES.

§ 299. Let there be given the series of square numbers,

1, 4, 9, 16, 25, &c.

If now we subtract the first of these numbers from the second, the second from the third, &c., we shall obtain what is called the *first order* of *differences*. If then we subtract the first of these *differences* from the second, &c., we shall obtain the *second order* of differences, and so on.

Thus,	1, 4, 9, 16, 25, 36, 49	
	3, 5, 7, 9, 11, 13	the first order of differences.
	2, 2, 2, 2, 2,	second " "
	0, 0, 0, 0,	third " "

What are the several orders of differences of the numbers, 1, 4, 10, 20, 35, 56, &c.?

1, 4, 10, 20, 35, 56	
3, 6, 10, 15, 21	the first order of diff.
3, 4, 5, 6	second "
1, 1, 1	third "
0, 0	fourth "

§ 300. Let there be an increasing series,

a_1, a_2, a_3, a_4, &c. Then we have

a_2-a_1, a_3-a_2, a_4-a_3, &c., first order of diff.

$a_3-2a_2+a_1$, $a_4-2a_3+a_2$, &c., second "

$a_4-3a_3+3a_2-a_1$, &c., third "

&c.

If we represent the first terms of these successive orders by D_1, D_2, D_3, D_4, &c. we shall have

$$D_1 = a_2 - a_1;$$
$$D_2 = a_3 - 2a_2 + a_1;$$
$$D_3 = a_4 - 3a_3 + 3a_2 - a_1;$$
$$D_4 = a_5 - 4a_4 + 6a_3 - 4a_2 + a_1;$$

and, obviously,

$$D_n = a_{n+1} - na_n + \frac{n(n-1)}{1.2} a_{n-1} - \frac{n(n-1)(n-2)}{1.2.3} a_{n-2} + \text{\&c.}; \quad (1)$$

the coefficients of a_{n+1}, a_n, &c., being the coefficients of the nth power of $a-x$.

Or, reversing the order of the terms,

$$D_n = \pm a_1 \mp na_2 \pm \frac{n(n-1)}{1.2} a_3 \mp \frac{n(n-1)(n-2)}{1.2.3} a_4 \pm \text{\&c.}; \quad (2)$$

taking the *upper* signs throughout, when n is *even;* and the *lower* signs, when n is *odd.*

Hence the first terms of the several orders of differences may be found without finding the remaining terms.

1. What is the first term of the third order of differences of the series, 1, 3, 6, 10, 15?

Here $a_1 = 1$, $a_2 = 3$, $a_3 = 6$, $a_4 = 10$, and $n = 3$.

$\therefore \; D_3 (= a_4 - 3a_3 + 3a_2 - {}_1a) = 10 - 3\times 6 + 3\times 3 - 1 = 0.$

So we should have $D_2 = a_1 - 2a_2 + a_3 = 1 - 2\times 3 + 6 = 1.$

2. Given the series 1, 8, 27, 64, 125, to find D_1, D_2, D_3 and D_4.

Ans. $D_1 = 7$, $D_2 = 12$, $D_3 = 6$, and $D_4 = 0$.

§ 301. From the values of D_1, D_2, &c. in § 300 we have

$$a_2 = a_1 + D,$$
$$a_3 = a_1 + 2D_1 + D_2,$$
$$a_4 = a_1 + 3D_1 + 3D_2 + D_3.$$

and, obviously,

$$a_n = a_1 + (n-1)D_1 + \frac{(n-1)(n-2)}{1.2} D_2 +$$
$$\frac{(n-1)(n-2)(n-3)}{1.2.3} D_3 + \frac{(n-1)(n-2)(n-3)}{1.2.3} D_3 + \text{\&c.}; \quad (2)$$

the coefficients of the terms being the coefficients of the $(n-1)$th power of $a+x$.

1. What is the fourth term of the series of squares,

$$1, 4, 9, 16, \&c.?$$

Here $a_1=1$, $D_1=3$, $D_2=2$, $D_3=0$, and $n=4$.

$\therefore a_4(=a_1+3D_1+3D_2+D_3)=1+3\times3+3\times2+0=16.$

2. What is the twentieth term of the same series?

Ans. 400.

3. What is the nth term of the same series?

$$a_n=a_1+(n-1)D_1+\frac{(n-1)(n-2)}{1.2}D_2+\&c.$$
$$=1+3(n-1)+(n-1)(n-2).$$
$$=1+3(n-1)+n(n-1)-2(n-1).$$
$$=1+n-1+n(n-1)=n^2.$$

4. What is the nth term of the series,

$$a, a+D, a+2D, a+3D, \&c.?$$

Ans. $a+(n-1)D$.

NOTES. (1.) The problem contained in the last example has been already considered (§ 250). In fact, the whole subject of equidifferent series, there treated, is only a particular case of the more general subject of *differences;* viz. the case, in which the first differences are constant (§ 249); and, of course, the second, and all higher differences are equal to zero.

(2.) It is proper to remark here, that an *equidifferent* series, having its *first* differences constant, is called a *difference* series of the *first order;* a series whose *second* differences are constant, is said to be of the *second* order; and so on. Thus, we have the series,

1, 2, 3, 4, 5, 6,	of the first order.
1, 3, 6, 10, 15, 21,	second "
1, 4, 10, 20, 35, 56,	third "

(3.) These, which are only particular examples of the various orders of difference series, have also this property; viz. the nth term of each series is equal to the $(n-1)$th term of the same series *plus* the nth term of the preceding series. And, consequently, the nth

term of each series is equal to the *sum* of n terms of the preceding series.

(4.) The numbers of the *second* series above (Note 2) are called *triangular* numbers; because the number of spherical bodies, as cannon balls, expressed by each of them, can be arranged in the form of an equilateral *triangle;* the number of balls on each side being expressed by the corresponding term of the natural series, 1, 2, 3, &c. So the numbers of the *third* series are called *pyramidal* numbers; because the number of balls expressed by each of them can be piled in a triangular *pyramid*, or tetraedron; the number of balls in the lowest course being expressed by the corresponding term of the triangular series, and the numbers in the other courses, by the preceding terms of that series. These, and other similar sets of numbers, are also called *figurate* numbers.

5. What is the fifteenth term of the series, 1^3, 2^3, 3^3, 4^3, &c.? *Ans.* 3375.

6. What is the nth term of the series, 1, 3, 6, 10, &c.? *Ans.* $\frac{n(n+1)}{2}$.

Let $n=1, 2, 3, 4, \ldots 10, 11, 12$, &c.; $0, -1, -2, -3$, &c. Also let $n=0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}$, &c.; and, finding the successive differences, see if the series is still of the second order.

7. What is the nth term of the series, 1, 4, 10, 20, 35, 56, &c.? *Ans.* $\frac{n(n+1)(n+2)}{2 \,.\, 3}$.

Let $n=1, 2, 3, 4, 5, 6, 7, 8$, &c.; $0, -1, -2, -3, -4$, &c. Also let $n=\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}$; and find the successive differences of the terms, and the *order* of the series (Note 2).

Note. (5.) The terms, whose places are denoted by 0, and -1 in the series of example 6th, and by 0, -1, and -2 in the series of example 7th, will be found to be each equal to 0. This will be illustrated by taking the succession of differences (as of the terms enclosed in brackets below), till they become constant, and continuing the constant differences, and, by means of them, the previous differences, and the given series backward. Thus,

$-5.$	$-4.$	$-3.$	$-2.$	$-1.$	0.	1.	2.	3.	4.	5.	6.
$-10,$	$-4,$	$-1,$	0,	0,	0,	[1,	4,	10,	20,	35,]	56,
	6,	3,	1,	0,	0,	1,	[3,	6,	10,	15,]	21,
		$-3,$	$-2,$	$-1,$	0,	1,	2,	[3,	4,	5,]	6,
			1,	1,	1,	1,	1,	1,	[1,	1,]	1,

In the corresponding series of the *fourth* order, 1, 5, 15, 35, 70, 126, we should find *four* terms equal to zero, and the terms, corresponding to the negative *local*[z] indices beyond, positive; and so on, the number of terms each equal to zero being equal to the number of the *order*; and the terms, corresponding to the negative indices beyond, being *positive* or *negative* according as the number of the *order* is *even* or *odd*.

§ 302. The formula (2) of the preceding section had primary reference to those terms only whose place in the series is expressed by whole numbers; i. e. to those denoted by *integral local indices.* We have found, however, by taking $n = \frac{1}{2}, \frac{3}{2}$, &c. in the general solution of examples 6th and 7th, terms corresponding to those *fractional* local indices, and still conforming to the general law of the series.

We shall find, in like manner, that the above formula applies in general to such *intermediate* terms corresponding to *fractional* local indices, equally as to terms whose local indices are integral; only giving a suitable value to n (§ 263).

Note. This is simply a more general form of the problem of *interpolation*; and applies to all series, whose differences of any order become either *zero*, or so small that they may be neglected.

1. Given $2^2 = 4$, $3^2 = 9$, and $4^2 = 16$; to find $(2\frac{1}{2})^2$.

Here $a_1 = 4$, $D_1 = 5$, $D_2 = 2$, $D_3 = 0$; and $n = 1\frac{1}{2}$.

$$\therefore \qquad a_n = a_{1\frac{1}{2}} = 4 + \tfrac{1}{2}\times 5 - \tfrac{1}{8}\times 2 = 6\tfrac{1}{4}.$$

2. Given $(2500)^{\frac{1}{2}} = 50$, $(2501)^{\frac{1}{2}} = 50.009,999,8$, $(2502)^{\frac{1}{2}} = 50.019,999,6$; to find $(2500.5)^{\frac{1}{2}}$.

Here $a_1 = 50$, $D_1 = .009,999,8$, $D_2 = 0$; and $n = 1\frac{1}{2}$.

$$\therefore \; a_n = (2500.5)^{\frac{1}{2}} = 50 + \tfrac{1}{2}\times .009,999,8 = 50.004,999,9\&c.$$

3. Given $64^{\frac{1}{2}} = 8$, $66^{\frac{1}{2}} = 8.124,038$, $68^{\frac{1}{2}} = 8.246,211$, and $70^{\frac{1}{2}} = 8.3666$; to find $65^{\frac{1}{2}}$.

(z) Lat. locus, *place.* Local indices *indicate* the *place* of the term in the series.

Here $a_1 = 8$, $D_1 = .124,038$, $D_2 = -.001,865$, $D_3 = \frac{1}{2}(.000,076 + .000,081) = .000,078$; and $n = 1\frac{1}{2}$.

Ans. $(65)^{\frac{1}{2}} = 8.062,257$.

4. Interpolate 3 terms between the fourth and fifth terms of the series,

4, 8, 12, 16, 20.

Here $a_1 = 4$, $D_1 = 4$, $D_2 = 0$; and $n = 4\frac{1}{4}, 4\frac{1}{2}, 4\frac{3}{4}$.

$$\therefore \qquad a_n = 4 + 3\tfrac{1}{4} \times 4 = 17;\ \&c.$$

Or $a_1 = 16$, $D_1 = 4$, $D_2 = 0$; and $n = 1\frac{1}{4}, 1\frac{2}{4}, 1\frac{3}{4}$.

$$\therefore \qquad a_n = 16 + \tfrac{1}{4} \times 4 = 17;\ \&c.$$

Ans. 17, 18 and 19. See § 263.

5. We find, in a table of natural sines,

$\sin 30° = .5$, $\sin 30°\ 10' = .502,517$,

$\sin 30°\ 20' = .505,030$, $\sin 30°\ 30' = .507,538$.

What is the sine of $30°\ 1'$? of $30°\ 2'$? of $30°, 3'$? of $30°\ 4'$?

Ans. $\sin 30°\ 1' = .500,252$; $\sin 30°\ 2' = .500,504$;

$\sin 30°\ 3' = .500,756$; $\sin 30°\ 4' = .501,007$.

§ 303. *a.*) In finding a term of the series by § 301, n being a whole number, the formula (2) will always terminate, because the coefficient $n(n-1) \ldots\ldots (n-n) = 0$. But, in interpolation (§ 302), the formula will not terminate, unless we find an order of differences equal to zero. For n being fractional, none of the factors, $n-1$, $n-2$, &c., can become *zero*; but they will become *negative*, and then increase numerically (§ 295. *d*). In this case, the required term can be found only by an infinite series.

b.) It will have been observed, that we have found terms, whose places are expressed both by integral and by fractional local indices, without knowing the law of the series into which they are introduced; knowing, in fact, nothing of the series but a few terms; or even a single term with the successive differences.

c.) Hence, obviously, the differences, together with a single term, determine the character of the series. They enable us to *continue* the series to any extent (§ 301), to supply *intermediate* terms (§ 302), and, as we shall see (§ 304), to find the *sum* of any number of terms.

§ 304. Let it be required to find the *sum* of n terms of the series,

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \ldots\ldots a_n.$$

Assume a series, whose first differences shall be the terms of the given series. Thus,

$$0, \quad a_1, \quad a_1+a_2, \quad a_1+a_2+a_3, \ldots a_1+a_2+a_3 \ldots +a_n.$$

Now the $(n+1)$th *term* of this last series is, evidently, the *sum* of n terms of the given series; and the $(n+1)$th differences of the last series are the nth differences of the given series.

Hence, marking the terms and differences of the assumed series with the accent $'$, we have, in formula (2) of § 301,

$$a'_1=0, \quad D'_1=a_1, \quad D'_2=D_1, \text{ \&c.};$$

and, putting $n+1$ in place of n, and denoting by S the required sum of n terms of the given series (i. e. the $(n+1)$th term, a'_{n+1}, of the assumed series), we find

$$S(=a'_{n+1})=na_1+\frac{n(n-1)}{1.2}D_1+\frac{n(n-1)(n-2)}{1.2.3}D_2+$$
$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}D_3+\text{ \&c.} \qquad (3)$$

1. What is the sum of n terms of the series,

1, 2, 3, 4, 5, 6, &c.?

Here $a_1=1$, $D_1=1$, and $D_2=0$.

$$\therefore \qquad S=n+\frac{n(n-1)}{1.2}=\frac{n(n+1)}{2}. \quad \text{See § 256. 3.}$$

2. What is the sum of n terms of the series, a, $a+D$, $a+2D$, &c. (i. e. an equidifferent series)?

Ans. $na+\frac{1}{2}n(n-1)D$. See §§ 253; 301. N. 1.

3. What is the sum of n terms of the series,

1, 3, 6, 10, 15, 21, 28, &c.?

Here $a_1=1$, $D_1=2$, $D_2=1$, and $D_3=0$.

$$\therefore \quad S=n+n(n-1)+\frac{n(n-1)(n-2)}{1.2.3}=\frac{n(n+1)(n+2)}{1.2.3}.$$

4. What is the sum of n terms of the series,

1, 4, 10, 20, 35, &c.?

Ans. $\dfrac{n(n+1)(n+2)(n+3)}{1.2.3.4}$.

5. What is the sum of n terms of the series,

1, 3, 5, 7, 9, 11, 13, &c.?

Ans. n^2. See § 256. 5.

6. What is the sum of n terms of the series,

1^2, 2^2, 3^2, 4^2, 5^2, &c.?

Ans. $\dfrac{n(n+1)(2n+1)}{1.2.3}$.

7. What is the sum of n terms of the series,

1^3, 2^3, 3^3, 4^3, 5^3, &c.?

Ans. $\dfrac{n^2(n+1)^2}{4}=\left(\dfrac{n(n+1)}{2}\right)^2$.

NOTE. From the result of examples 1st and 7th, we have

$$1^3+2^3+3^3\,..+n^3=(1+2+3\,..+n)^2.$$

CHAPTER XIV.

INFINITE SERIES.

§ 305. An INFINITE SERIES, we have seen, may arise from an imperfect division (§ 87. *a*); or from the extraction of a root of an imperfect power (§ 170. N. 5); or by the continuation of an equimultiple (§ 261. *c*) series to infinity.

Infinite series of various forms are also developed by the method of undetermined coefficients (§ 281), and by the binomial Theorem (§ 295. *d*); and by many other processes, which we are not yet prepared to investigate, and some of which are beyond the reach of elementary Algebra.

§ 306. As the processes of developing infinite series are so various, the methods of summing them are equally various. Even of those which are summed by the elementary processes of Algebra, we shall consider here only one or two of the simplest.

a.) The method of summing a *converging* infinite *equimultiple* series has already been investigated (§ 261. *c*).

b.) The true sum of an infinite series resulting from division, or from the development of a fraction by undetermined coefficients, is the fraction from whose development the series originated; and this, whether the series be converging or diverging (§ 87. *d*. *f*).

We may, moreover, approximate to the value of a converging series by the actual addition of a small number of

the terms (smaller or greater, according to the greater or less rapidity of the convergence).

But the doctrine of infinite series proposes to find convenient expressions for the sum of any part, or the whole of a series, without the labor of adding the several terms.

§ 307. We have $\frac{q}{m}-\frac{q}{m+p}=\frac{pq}{m(m+p)}$. § 118.

$\therefore$ (§ 42. *d*) $\frac{q}{m(m+p)}=\frac{1}{p}\left(\frac{q}{m}-\frac{q}{m+p}\right)$. (1) That is,

A fraction of the form $\frac{q}{m(m+p)}$ is equal to $\frac{1}{p}$ of the *difference* between the two fractions $\frac{q}{m}$ and $\frac{q}{m+p}$. Now, as this is true of *any* fraction of this form, it is true of *each of the terms of a series composed of such fractions.* Hence the sum of such a series will be equal to $\frac{1}{p}$ of *the difference between two series*, one consisting of terms of the form $\frac{q}{m}$, and the other, of the form $\frac{q}{m+p}$.

1. Let it be required to find the sum of the series,

$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$&c., to n terms, and also to infinity.

Here we have

$m(m+p)=1.2,\quad 2.3,\ \&c.=1(1+1),\ 2(2+1),\ \&c.$

$\therefore\quad q=1,\quad p=1,\quad$ and $m=1, 2. 3 \,.\,.\, n$, successively.

Represent also the sum of n terms by S_n, and, by analogy, the sum of an infinite number of terms by S_∞. Then

$$S_n=\left.\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ \ldots\ +\frac{1}{n}\\ \quad-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\ \ldots\ -\frac{1}{n}-\frac{1}{n+1}\end{array}\right\}=1-\frac{1}{n+1}=\frac{n}{n+1}$$

If $n=\infty$, we have (§ 138) $\frac{1}{n+1}=0$; and $\therefore S_\infty=1$.

Otherwise; when $n=\infty$, we have (§ 261. N. 2)

$$\frac{n}{n+1}=\frac{n}{n}=1;\quad \text{and } \therefore S_\infty=1.$$

2. What is the sum of the series, $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+$ &c., to n terms, and also to infinity?

Here $\frac{1}{1.3}\left(=\frac{1}{1(1+2)}\right)$, $\frac{1}{3.5}\left(=\frac{1}{3(3+2)}\right)$, &c. are of the

form $$\frac{q}{(2n-1)[(2n-1)+2]}.$$

We have, therefore $q=1$, $p=2$, and $m=2n-1$.

$$\text{Hence, } S_n=\frac{1}{2}\left\{\begin{array}{l} 1+\frac{1}{3}+\frac{1}{5}\;\ldots\ldots+\frac{1}{2n-1} \\ \quad-\frac{1}{3}-\frac{1}{5}\;\ldots\ldots-\frac{1}{2n-1}-\frac{1}{2n+1}\end{array}\right\}.$$

$$\therefore\qquad S_n=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{n}{2n+1}.$$

Also, making $n=\infty$, we have

$$\frac{1}{2n+1}=0, \text{ or } \frac{n}{2n+1}=\frac{n}{2n};\quad \text{and } \therefore S_\infty=\frac{1}{2}.$$

3. Find the sum of n terms of the series,

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\text{\&c.};$$

and also of the whole series to infinity.

Here $\frac{1}{1(1+3)}$, $\frac{1}{2(2+3)}$, &c. give,

$$q=1, p=3, \text{ and } m=1, 2, 3\;\ldots\ldots n.\qquad\therefore$$

$$S_n=\frac{1}{3}\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots+\frac{1}{n} \\ \qquad\qquad-\frac{1}{4}\ldots-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\end{array}\right\}.$$

$$S_n = \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right).$$

$$= \frac{1}{3}\left(\frac{n}{n+1}+\frac{n}{2n+4}+\frac{n}{3n+9}\right).$$

$$= \frac{n}{3n+3}+\frac{n}{6n+12}+\frac{n}{9n+27}.$$

Also $S_\infty = \frac{1}{3}(1+\frac{1}{2}+\frac{1}{3}) = \frac{11}{18}$.

4. Find the sum of the series, $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$&c.; (the denominators being the terms of a differential series of the second order, viz. the triangular numbers;).

Dividing the series by 2, we have

$$\frac{S_\infty}{2} = \frac{1}{2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\frac{1}{4\,.\,5}+\text{\&c.} = 1,$$

by example 1st, above.

$$\therefore \qquad S_\infty = 2.$$

5. What is the sum of the series,

$$\frac{1}{3\,.\,8}+\frac{1}{6\,.\,12}+\frac{1}{9\,.\,16}+\text{\&c.}?$$

Take out of each term the common factor $\frac{1}{12}$, by dividing the second factors of the denominator by 4, and the first factors by 3.

Ans. $S_\infty = \frac{1}{12}$.

6. Sum the series, $\frac{2}{3\,.\,5}-\frac{3}{5\,.\,7}+\frac{4}{7\,.\,9}-\frac{5}{9\,.\,11}+$&c.

Here $p=2$, $q=n+1$, and $m=2n+1$; n being taken $=1, 2, 3,$ &c., successively.

$$\therefore \quad S_n = \frac{1}{2}\left\{\begin{array}{l} \frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}+\ldots\mp\frac{n+1}{2n+1} \\ \quad -\frac{2}{5}+\frac{3}{7}-\frac{4}{9}+\ldots\mp\frac{n}{2n+1}\pm\frac{n+1}{2n+3} \end{array}\right\}.$$

Note. If n is *infinite*,

$$1-1+1-1+\text{\&c.} = \frac{1}{1+1} = \frac{1}{2}.$$

The sum of n terms of this series will be equal to 1, if n is *finite* and *even*; to 0, if it is finite and *odd*. Hence we have

$$S_\infty = \tfrac{1}{2}[\tfrac{2}{3}-(1-1+1-1+\&c.)] = \tfrac{1}{2}(\tfrac{2}{3}-\tfrac{1}{2}) = \tfrac{1}{12}.$$

$$S_n = \frac{1}{2}\left(\frac{2}{3}-\frac{1}{2}\mp\frac{1}{2}\pm\frac{n+1}{2n+3}\right) = \frac{1}{2}\left\{\frac{1}{6}\mp\left(\frac{1}{2}-\frac{n+1}{2n+3}\right)\right\}.$$

$$= \frac{1}{2}\left(\frac{1}{6}\mp\frac{1}{2(2n+3)}\right) = \left(\frac{1}{12}\mp\frac{1}{4(2n+3)}\right).$$

7. What is the sum of the series,

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\&c.?$$

Ans. $S_n = \dfrac{3}{4}-\dfrac{1}{2}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}\right)$; $S_\infty = \dfrac{3}{4}$.

8. What is the sum of the series,

$$\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\&c.?$$

Ans. $S_\infty = \dfrac{1}{4}$.

9. What is the sum of the series,

$$\frac{4}{1.5}+\frac{4}{5.9}+\frac{4}{9.13}+\frac{4}{13.17}+\&c.? \quad \textit{Ans. } S_\infty = 1.$$

Here $q=4$, $p=4$, and $m=4n+1$; n being taken $=0, 1, 2, 3$, &c., successively.

§ 308. Again, as we evidently have (§ 118)

$$\frac{1}{rp}\left\{\frac{q}{m(m+p)\,..\,(m+(r-1)p)}-\frac{q}{(m+p)(m+2p)\,..\,(m+rp)}\right\}$$

$$= \frac{q}{m(m+p)(m+2p)\;.\;.\;.\;(m+rp)}, \qquad (2)$$

a series of terms in the form of the *second* member of this equation is equal to $\dfrac{1}{rp}$ of the difference of two series of terms in the form of those in the second factor of the *first* member.

1. Sum the series, $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+\&c.$

Here $q=n+3, p=1, r=2$, and $m=n=1, 2, 3,$ &c.

$$\therefore S_\infty=\frac{1}{2}\left\{\begin{array}{l}\frac{4}{1.2}+\frac{5}{2.3}+\frac{6}{3.4}+\&c.\\ \qquad -\frac{4}{2.3}-\frac{5}{3.4}-\&c.\end{array}\right\}=$$

$$\frac{1}{2}\left(\frac{4}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\&c.\right)=\frac{5}{4},$$ by § 307. 1.

2. Given $\frac{1}{1.3.5}+\frac{4}{3.5.7}+\frac{7}{5.7.9}+\&c.$, to find S_∞.

Ans. $S_\infty=\frac{5}{24}$.

3. Given $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\&c.$, to find S_∞.

Ans. $S_\infty=\frac{1}{18}$.

4. Given $\frac{1}{1.3.5.7}+\frac{1}{3.5.7.9}+\frac{1}{5.7.9.11}+\&c.$, to find S_∞.

Ans. $S_\infty=\frac{1}{72}$.

5. Given $\frac{6^2}{1.2.3.4}+\frac{7^2}{2.3.4.5}+\frac{8^2}{3.4.5.6}$, to find S_∞.

Ans. $S_\infty=\frac{89}{36}$.

CHAPTER XV.

LOGARITHMS.

§ 309. All finite, positive numbers may be regarded as *powers* of any finite, positive number except unity.

Thus, if 10 be taken as the *base* (§ 22. N.), 1, 10, 100, 1000, &c., $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c., will be expressed as *integral* powers of the base; those *above* 1, *positive;* those *below*, *negative.*

Moreover, it is obvious, that all numbers *between* the integral powers can be expressed as *fractional* powers, either positive or negative. That is, the base can be separated into factors so small, that a certain number of them multiplied together (§ 12), or divided out of unity (§ 14), shall produce, at least to any degree of approximation, any given number (§ 319).

a.) It is evident that 1 cannot be taken as a base of such a system of powers, because every power of 1 is 1.

b.) It is also evident, that, if a *proper fraction,* as $\frac{1}{10}$, be taken for the base, *fractions* will be expressed as *positive,* and *integers,* as *negative* powers.

c.) The base must be a *positive* number; for if it were negative, only such positive numbers could be expressed as should coincide with its even powers; and only such negative numbers, as should coincide with its odd powers.

d.) Again, of a positive base no *negative* number can be a *power*, unless the denominator of its exponent be even, and the numerator odd (§§ 11. N. 2; 23. *e, f*). Hence the limitation to *positive* numbers.

§ 310. If all numbers, with the limitations above explained, were thus expressed as powers of a single number, the labor of *multiplication* and *division* would obviously be reduced to the *adding* and *subtracting* of the *exponents* (§§ 15, 16).

Thus, since $100 = 10^2$, and $1000 = 10^3$;

$$\therefore \quad 100 \times 1000 = 10^2 \times 10^3 = 10^5 = 100\,000.$$

Also, $\quad 1000 = 10^3, \quad \frac{1}{100} = 10^{-2}.$

$$\therefore \quad 1000 \div \tfrac{1}{100} = 10^3 \div 10^{-2} = 10^5 = 100\,000.$$

$$2 = 10^{\frac{301030}{1000000}}, \quad 5 = 10^{\frac{698970}{1000000}}.$$

$$\therefore \quad 2 \times 5 = 10^{.301030} \times 10^{.698970} = 10^1.$$

§ 311. When numbers are thus expressed as powers of another number, the *exponents of those powers* are called LOGARITHMS[a] *of the numbers* so expressed; and the number whose powers are thus employed, is called the BASE (§ 22. N.), and sometimes also the RADIX (§ 23. *d*), of the system.

Hence, for a given base,

§ 312. *The logarithm of any number is the exponent of the power to which the base must be raised, to produce that number.*

Thus, 2 is the logarithm of 100 to the base 10; because 2 is the exponent of the power to which 10 must be raised to produce 100.

So, because $2 = 10^{.301030}$, .301 030 is the logarithm of 2 to the base 10; for .301 030 is the exponent of the power to which the base, 10, must be raised to produce 2.

(*a*) Gr. λόγος, *ratio*, ἀριθμὸς, *number*; number of the ratio.

a.) Cor. I. *The logarithm of the base of the system is always* 1. § 11. *a.*

b.) Cor. II. *The logarithm of unity in every system is zero.* § 13.

§ 313. The base of the system of logarithms in common use is 10. We have, therefore,

$$\log 10{,}000 = \log 10^4 = 4, \quad \log 1000 = \log 10^3 = 3,$$
$$\log (100 = 10^2) = 2, \quad \log (10 = 10^1) = 1,$$
$$\log (1 = 10^0) = 0, \quad \log (\tfrac{1}{10} = 10^{-1}) = -1,$$
$$\log (.01 = 10^{-2}) = -2, \quad \log (.001 = 10^{-3}) = -3, \text{ \&c.}$$

a.) Hence, obviously, the common logarithm of any number between 1 and 10 is a proper fraction; that of any number between 10 and 100 is $1 +$ a fraction; between 100 and 1000, it is $2 +$ a fraction; &c.

b.) Again, the common logarithm of any number between $\frac{1}{10}$ and 1, as .3454, is between -1 and 0, and therefore it is $-1 +$ a fraction; of a number between .01 and .1, as .0205, the logarithm is $-2 +$ a fraction; of a number between .001 and .01, the logarithm is $-3 +$ a fraction.

§ 314. *c.*) *The integral part of a common logarithm is* called its CHARACTERISTIC; because it characterizes the logarithm by showing, where in the series of the powers of 10 the number of which it is the logarithm falls. The characteristic of the logarithm of a number *greater than ten* is *positive;* of a number *less* than unity, *negative* (§ 309).

d.) Moreover (§ 313), the characteristic of the common logarithm of any number is always equal to the *exponent of the integral power of* 10 *next below* that number; and hence, in the common system,

(1.) If a number be *greater than unity, the characteristic* of its logarithm is *one less than the number of its integral places;* (2.) if *less than unity,* the *negative* characteristic

is numerically *one greater than the number of cyphers between the decimal point and the first significant figure on the left, in the decimal expression* of the fraction.

e.) Otherwise; *the characteristic* of a logarithm of a number is equal to the *number of places from the unit place to the highest significant figure, including the latter; positive,* if that figure be on the *left* of the unit place; *negative,* if on the *right.*

§ 315. *f.*) We have $\log 20 = \log 10 + \log 2 = 1 + \log 2$;

$$\log 200 = \log 100 + \log 2 = 2 + \log 2;$$

$$\log 525 = \log 10 + \log 52.5 = 1 + \log 52.5;$$

$$= \log 100 + \log 5.25 = 2 + \log 5.25.$$

So $\log .525 = \log 525 - \log 1000 = -3 + \log 525.$

But adding *whole units* to a mixed number cannot affect its *fractional* part. Hence, the *decimal* part of the common logarithm corresponding to a number expressed by any figures whatever, *is the same, whether those figures stand all on the right, or a part or all on the left of the decimal point.* Thus, we have

$$\log 25 = 1.397\,960; \quad \log 250 = 2.397\,960;$$

$$\log 25{.}000 = 4.397\,960; \quad \log .025 = -2.397{.}960.$$

g.) The principles of §§ 313–315 result from the employment of the *base of our scale of notation* as base of the system of logarithms. On account of this peculiarity, the *common,* or *Briggs's*[b] logarithms are much more convenient than any other for numerical computations; and are, therefore, in universal use for that purpose.

§ 316. The following principles, resulting from the nature of logarithms as *exponents* (§§ 309–312), are formally stated here for reference.

(*b*) So called from Mr. Henry Briggs, who first suggested to Lord Napier, the inventor of logarithms, the employment of 10 as a base; and who completed the computation of the first table of logarithms with that base.

1. *The sum of the logarithms of any number of factors is equal to the logarithm of their product* (§ 15).

2. *The logarithm of a dividend minus the logarithm of a divisor, is equal to the logarithm of their quotient* (§ 16).

3. *The double* of the logarithm of a number is equal to the logarithm of the *square* of that number; the *triple* of its logarithm, to the logarithm of its *cube*, &c.; the *half*, to the logarithm of its *square root; one third*, to the logarithm of its *cube root*, &c.; and, in general, *n times the logarithm of a number is equal to the logarithm of the nth power of the number* (whether n be integral or fractional, positive or negative). See § 24. *d.*

§ 317. It is evident, that *if a set of numbers* form an *equimultiple* (§ 257) series, *their logarithms* will form an *equidifferent* (§ 249) series.

Thus, the logarithms of 1, 10, 100, 1000, are 0, 1, 2, 3.

So the logarithms of a, am, am^2, &c., form an equidifferent series, of which the common difference is the logarithm of m. Hence,

§ 318. If between two numbers we interpolate any number of equimultiple means (§ 265), and between the corresponding logarithms interpolate the same number of equidifferent means (§ 255), these last terms will form the *logarithms* of the several terms of the first series. Thus,

The equimultiple mean between 1 and 10 = 3.162 277 7.

The equidifferent mean between 0 and 1 = $\frac{1}{2}$.

$$\therefore \qquad \log 3.162\,277\,7 = \tfrac{1}{2}.$$

§ 319. If the base, 10, be separated into 1 000 000 equal factors, 301 030 of these factors multiplied together will, within an extremely small fraction, produce 2; in like manner, 477 121 will produce 3; 602 060 will produce 4; 500 000 will produce 3.162 277 7; and so on. Hence we have

$$\log 2 = .301\,030; \quad \log 3 = .477\,121;$$
$$\log 4 = .602\,060; \quad \log 3.162\,277\,7 = .500\,000.$$

If, indeed, instead of taking the mean between 10 and 3.162 277 7, we had taken the mean between 1 and 3.162 277 7 (that is, if we had taken the square root of 3.162 277 7), we should have separated 10 into its four equal factors, one of which would be the number whose logarithm is $\frac{1}{4}$. A third extraction of the square root would give us one of its eight equal factors; a fourth, one of its sixteen; a fifth, one of its thirty-two equal factors; and so on.

Continuing this process, the twentieth extraction of the square root would separate the base, 10, into more than a million equal factors (1 048 576). Consequently the logarithm of one of these factors must be

$$\frac{1}{1048576} = .000\,000\,954.$$

If now we *multiply* together a number of these factors sufficient to produce 2, 3, 4, &c., and *add* together their logarithms (i. e. as the logarithms of the equal factors are, of course, equal, if we multiply the logarithm of one of these factors by the number of the factors), the sum of these logarithms will be the logarithm of the number produced by the combination of the factors. A combination of 315 545 of these equal factors will approximately produce 2. Hence we have

$$\log 2 = 315\,545 \times .000\,000\,954 = .301\,030.$$

§ 320. Let $$10^x = 3. \qquad (1)$$

That is, let $$x = \log 3.$$

To find x, put equation (1) under the form,

$$[1+(10-1)]^{tx} = [1+(3-1)]^t; \qquad (2)$$

which is evidently true, whatever be the value of t. Then

$$1+tx(10-1)+\frac{tx(tx-1)}{1\,.\,2}(10-1)^2+\frac{tx(tx-1)(tx-2)}{1\,.\,2\,.\,3}$$

$$(10-1)^3+\&c. = 1+t(3-1)+\frac{t(t-1)}{1\,.\,2}(3-1)^2+$$

$$\frac{t(t-1)(t-2)}{1\,.\,2\,.\,3}(3-1)^3+\&c. \qquad \text{§ 294.}$$

Or, canceling 1, and dividing by t,

$$x(10-1)+\frac{x(tx-1)}{1.2}(10-1)^2+\frac{x(tx-1)(tx-2)}{1.2.3}(10-1)^3$$

$$+\&c.=3-1+\frac{t-1}{1.2}(3-1)^2+\frac{(t-1)(t-2)}{1.2.3}(3-1)^3+\&c.$$

Developing the coefficients of 10—1 and 3—1, arranging according to the powers of t, and putting B, C, &c., B', C', &c. to represent the coefficients of t^1, t^2, &c. on the two sides, we have

$$x[10-1-\tfrac{1}{2}(10-1)^2+\tfrac{1}{3}(10-1)^3-\&c.]+Bt+Ct^2+\&c.$$
$$=3-1-\tfrac{1}{2}(3-1)^2+\tfrac{1}{3}(3-1)^3-\&c.+B't+C't^2+\&c.$$

Now this equation being true for all values of t, we have (§ 277)

$$x[10-1-\tfrac{1}{2}(10-1)^2+\tfrac{1}{3}(10-1)^3-\tfrac{1}{4}(10-1)^4+\&c.]$$
$$=3-1-\tfrac{1}{2}(3-1)^2+\tfrac{1}{3}(3-1)^3-\tfrac{1}{4}(3-1)^4+\&c.$$

$$\therefore\quad x=\log 3=\frac{3-1-\frac{1}{2}(3-1)^2+\frac{1}{3}(3-1)^3-\&c.}{10-1-\frac{1}{2}(10-1)^2+\frac{1}{3}(10-1)^3-\&c.}. \quad (3)$$

§ 321. But the denominator of this fraction is a *diverging* series; as is the numerator, unless the number whose logarithm is sought is very near unity. If, however, we take the nth root of both sides of equation (1), we shall have

$$10^{\frac{x}{n}}=3^{\frac{1}{n}};\quad \text{or } (10^{\frac{1}{n}})^x=3^{\frac{1}{n}}.$$

Then, as before,

$$(1+10^{\frac{1}{n}}-1)^{tx}=(1+3^{\frac{1}{n}}-1)^t;$$

and, developing as before, we have

$$x=\log 3=\frac{3^{\frac{1}{n}}-1-\frac{1}{2}(3^{\frac{1}{n}}-1)^2+\frac{1}{3}(3^{\frac{1}{n}}-1)^3-\&c.}{10^{\frac{1}{n}}-1-\frac{1}{2}(10^{\frac{1}{n}}-1)^2+\frac{1}{3}(10^{\frac{1}{n}}-1)^3-\&c.}. \quad (4)$$

§ 322. Now n may be any number whatever; but, in order that the series may converge properly, it must be *very*

large, and *positive*. Let it be taken so great, that $3^{\frac{1}{n}}$ and $10^{\frac{1}{n}}$ may each be expressed by $1+$ *a decimal fraction* whose first eight places are cyphers. Then we shall have

$10^{\frac{1}{n}}-1=$ a decimal whose first *eight* places are cyphers.

$\therefore$ $(10^{\frac{1}{n}}-1)^2=$ a decimal whose first *sixteen* places are cyphers.

Hence, the second, and, with still more reason, the subsequent terms of the series can have no effect on the first *fifteen* places of the denominator (i. e. on its first *seven significant* figures).

The first term, then, will give the value of the denominator correct to fifteen places of decimals; seven of which are significant.

The same reasoning will apply, with still greater force to the numerator; the first term of which will be a still closer approximation to the true value of the series.

Hence, n being very large, we shall have approximately,

$$x(=\log 3)=\frac{3^{\frac{1}{n}}-1}{10^{\frac{1}{n}}-1}. \qquad (5)$$

Let now $n=2^{60}$. Then we shall find $3^{\frac{1}{n}}$, and $10^{\frac{1}{n}}$ by extracting the square root of 3 and of 10 *sixty* times in succession, and we shall have

$$x=\log 3=\frac{3^{\frac{1}{2^{60}}}-1}{10^{\frac{1}{2^{60}}}-1}.$$

$$=\frac{.000\,000\,000\,000\,000\,000\,952\,894\,264\,074\,589}{.000\,000\,000\,000\,000\,001\,997\,174\,208\,125\,505}.$$

$\therefore$ $\qquad x=\log 3=.477\,121\,254\,719\,662,$&c.

NOTE. Briggs took $n=2^{54}$. A much smaller value of n would give results sufficiently accurate for all ordinary purposes. The process would still, however, be extremely laborious; and has been superseded by far more convenient and rapid processes. But the for-

mulæ, which we have obtained, furnish a convenient method of exhibiting some very important *properties* of logarithms.

§ 323. For greater convenience and clearness, we shall generalize equations (1), (2), (3), (4) and (5), by putting y in place of 3, and a in place of 10. Then we shall have

$$a^x = y \qquad (6); \quad \text{or } x = \log y;$$

$$(1+a-1)^{tx} = (1+y-1)^t; \tag{7}$$

$$x = \log y = \frac{y-1-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\&c.}{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}; \tag{8}$$

$$x = \log y = \frac{\sqrt[n]{y}-1-\frac{1}{2}(\sqrt[n]{y}-1)^2+\frac{1}{3}(\sqrt[n]{y}-1)^3-\&c.}{\sqrt[n]{a}-1-\frac{1}{2}(\sqrt[n]{a}-1)^2+\frac{1}{3}(\sqrt[n]{a}-1)^3-\&c.}; \tag{9}$$

and, n being very large, approximately (§ 322),

$$x = \log y = \frac{y^{\frac{1}{n}}-1}{a^{\frac{1}{n}}-1}. \tag{10}$$

§ 324. Now it is manifest, that, in each of the equations (8), (9) and (10), for a given value of n, [1.] the value of the *denominator* of the second member depends *solely on the base;* and, *for the same base* (i. e. in a *given system*), it remains *constant, whatever be the number whose logarithm is required;* [2.] the value of the *numerator* depends solely *on the number whose logarithm is sought,* and is, therefore, *constant for the same number in all systems* (i. e. *whatever be the base);* and [3.] the *denominator* is the *same function* of a as the *numerator* is of y.

§ 325. Again, representing the constant denominator by $f(a)$, we have from (8) of § 323

$$\log y = \frac{1}{f(a)}[y-1-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\&c.];$$

$$= \frac{1}{f(a)}f(y); \tag{11}$$

in which

$$f(a) = a-1-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\&c.; \tag{12}$$

and $f(y) = y-1-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\frac{1}{4}(y-1)^4+\&c.$

In like manner, in another system whose base is a', we shall have, obviously,

$$\log' y = \frac{1}{f(a')}[y-1-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\&c.] = \frac{1}{f(a')}fy.$$

Hence, $$\log(y) : \log'(y) = \frac{1}{f(a)} : \frac{1}{f(a')}. \qquad (13)$$

That is,

§ 326. *The logarithm of a number, taken in different systems, varies* (§ 245) *inversely as the function of the base.*

a.) If the base, a, be given, $f(a)$ can be found from formula (24); and, on the other hand, if $f(a)$ be given, a value can be found for the base, a, to correspond.

b.) Thus, Lord Napier, the inventor of logarithms, took $f(a)=1$, and constructed the first table of logarithms ever published, on that hypothesis. Consequently, denoting the Naperian logarithm by Log or L, and the logarithm in any other system by log or l, log$'$ or l', &c., we have

$$\text{Log}\, y = y-1-\tfrac{1}{2}(y-1)^2+\tfrac{1}{3}(y-1)^3-\&c. \qquad (14)$$

$\therefore$ from (11) $$\log y = \frac{1}{f(a)}\text{Log}\, y. \qquad (15)$$

That is,

§ 327. The *Naperian logarithm* of any number, *multiplied by* $\frac{1}{f(a)}$ gives the logarithm of that number in the *system whose base* is a.

§ 328. The quantity $\frac{1}{f(a)}$ is called the MODULUS[c] of the system whose base is a; because it *expresses or measures the ratio of any logarithm in that system, to the Naperian logarithm of the same number.*

Hence [§ 325. (13)],

a.) Cor. I. *The logarithms of any number in different systems are to each other as their moduli.* And hence,

(c) Lat., *a measure.*

b.) Cor. II. *The logarithms of any numbers in one system are to the logarithms of the same numbers in another system in a constant ratio*, viz. in the ratio of their moduli (§ 328. *a*). Thus,

$$l\,2 : l'\,2 = l\,3 : l'\,3 = le : l'e = la : l'a = M : M'$$

(M and M' representing the moduli).

c.) Also, since $ly = M . Ly$, $M = \frac{ly}{Ly} = \frac{la}{La} = \frac{le}{Le}$ &c. (16)

That is,

Cor. III. *The modulus of any system is equal to any logarithm in that system divided by the Naperian logarithm of the same number.*

d.) Again, since a is the base, $la = 1$.

$$\therefore \qquad M = \frac{1}{La}. \tag{17}$$

That is,

Cor. IV. The *modulus* of any system is *the reciprocal of the Naperian logarithm of the base of the system.*

Thus, the modulus of the common system is $\frac{1}{\text{Log } 10}$.

Hence (§ 328)

e.) We have $\qquad f(a) = \text{Log } a.$ (18)

That is,

The *denominator* of the second member of equation (8) in § 323 is the *Naperian logarithm of the base.*

Note. This is evident also from § 324. 3, and § 326. (14).

f.) Again, if $e =$ the Naperian base, we have $Le = 1$.

$$\therefore \qquad M = le. \tag{19}$$

That is,

Cor. V. The *modulus* of any system is equal to the *logarithm of the Naperian base, taken in that system.*

Thus, the modulus of the common system is the common logarithm of e.

§ 329. *g.*) $f(a) = La = nLa^{\frac{1}{n}}$. § 316. 3.

Also [§ 326. (14)]

$$La^{\frac{1}{n}} = a^{\frac{1}{n}} - 1 - \tfrac{1}{2}(a^{\frac{1}{n}}-1)^2 + \tfrac{1}{3}(a^{\frac{1}{n}}-1)^3 - \tfrac{1}{4}(a^{\frac{1}{n}}-1)^4 + \&c. \quad (20)$$

$$\therefore \quad f(a)\,[= La = nLa^{\frac{1}{n}}] = n[a^{\frac{1}{n}} - 1 - \tfrac{1}{2}(a^{\frac{1}{n}}-1)^2 + \&c.];$$

$$= n(a^{\frac{1}{n}} - 1), \quad (21)$$

when n is very large.

$$\therefore \quad M\left(= \frac{1}{f(a)} = \frac{1}{La} = \frac{1}{nL(\sqrt[n]{a})}\right) = \frac{1}{n(\sqrt[n]{a}-1)}.$$

$$= \frac{1}{n}.\frac{1}{a^{\frac{1}{n}}-1}. \quad (22)$$

And, for the common system, taking $n = 2^{60}$, we have approximately (21)

$$f(a) = L10 = nL10^{\frac{1}{n}} = 2^{60}.\,(10^{\frac{1}{2^{60}}} - 1).$$

$$\therefore \quad M = \frac{1}{L10} = \frac{1}{2^{60}}.\frac{1}{10^{\frac{1}{2^{60}}}-1}, \quad \text{or} \quad \frac{1}{2^{60}}.\frac{1}{\sqrt[2^{60}]{10}-1}.$$

Now we have

$$\frac{1}{2^{60}} = 0.000\,000\,000\,000\,000\,000\,867\,361\,737\,988\ \&c., \text{ and}$$

$$10^{\frac{1}{2^{60}}} = 1.000\,000\,000\,000\,000\,001\,997\,174\,208\,125\ \&c.$$

$$\therefore \quad M = \frac{867\,361\,737\,988\ \&c.}{1\,997\,174\,208\,125\ \&c.}.$$

$$\therefore \quad M\left(= \frac{1}{L10} = le\right) = 0.434\,294\,481\,903\,251\ \&c. \quad (23)$$

§ 330. *h.*) If we take $f(a) = 1$, then (§§ 326. *b*; 328. *f*) $a = e$, and [§ 329. (21)], approximately,

$$1 = n(e^{\frac{1}{n}} - 1); \quad \text{or } \frac{1}{n} = e^{\frac{1}{n}} - 1; \quad \text{or } 1 + \frac{1}{n} = e^{\frac{1}{n}}.$$

$$\therefore \quad e = \left(1 + \frac{1}{n}\right)^n. \quad \therefore (\S\ 294)$$

$$e = 1 + n\frac{1}{n} + \frac{n(n-1)}{1.2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{1}{n}\right)^3 + \&c.;$$

or, reducing, and neglecting all the terms, which have n, n^2, &c. in the denominator, inasmuch as (n being $= 2^{60}$) they cannot affect the first sixteen places of decimals,

$$e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. \quad (24)$$

$$\therefore \quad e = 2.718\,281\,828\,459\,045\ \&c. \quad (25)$$

Note. To find the sum of any number of terms of this series, divide the second term by 2 for the third; the third term by 3 for the fourth, and so on; and then add the terms. Thus,

	1
2)	1
3)	0.5
4)	0.1 666 666 666 666 666 666 666 666 666 666 666 &c.
&c.	0.0 416 666 666 666 666 666 666 666 666 666 666 &c.

§ 331. *i.*) If we take $f(a) = 1$, § 330.

we have $$M\left(= \frac{1}{f(a)} = \frac{1}{f(e)} = \frac{1}{Le}\right) = 1. \quad (26)$$

That is,

The modulus of the Naperian system is unity.

This is evident also from §§ 327, 328, c, d, e, or f.

Notes. (1). On account of this property the *Naperian* system is sometimes called the *natural* system; it being the standard to which all other systems are referred by their moduli. (2.) The Naperian system is also, in general, most convenient for algebraic (§ 315. g) investigations and expressions; because the modulus, being unity, need not be written. (3.) Naperian logarithms are also sometimes called *hyperbolic* logarithms, because they express certain areas connected with the hyperbola. This name, however, is less appropriate, because other systems of logarithms have the same property with reference to different forms of the hyperbola.

§ 332. We have [§§ 328; 325. (12)]

$$M = \frac{1}{f(a)} = \frac{1}{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}$$

$\therefore$, § 323. (8), $\log y = M[y-1-\frac{1}{2}(y-1)^2+\&c.]$. (27)

$\therefore \qquad \log y^{\frac{1}{n}} = M[y^{\frac{1}{n}}-1-\frac{1}{2}(y^{\frac{1}{n}}-1)^2+\&c.]$; (28)

or $\qquad \log y^{\frac{1}{n}} = M(y^{\frac{1}{n}}-1)$, (29)

when n is very large.

Also (§ 327) $\quad \log y = M \operatorname{Log} y$. (30)

§ 333. Let $a^{\frac{x}{n}} = y^{\frac{1}{n}}$, n being a very large positive number, as 2^{60}, and y being a small number greater than unity; so that $y^{\frac{1}{n}}$ may differ but *very little* from *unity*, and $\frac{x}{n}$, from *zero*. Then [§§ 312; 332. (30), (29)]

$$\frac{x}{n} = \log y^{\frac{1}{n}} = M . \operatorname{Log} y^{\frac{1}{n}} = M(y^{\frac{1}{n}}-1).$$

So $\quad \frac{x'}{n} = ly'^{\frac{1}{n}} = M(y'^{\frac{1}{n}}-1)$; $\frac{x''}{n} = ly''^{\frac{1}{n}} = M(y''^{\frac{1}{n}}-1)$; &c.

$$\therefore \qquad ly^{\frac{1}{n}} : ly'^{\frac{1}{n}} : ly''^{\frac{1}{n}} = y^{\frac{1}{n}}-1 : y'^{\frac{1}{n}}-1 : y''^{\frac{1}{n}}-1.$$

or $\qquad \log y^{\frac{1}{n}} \doteq y^{\frac{1}{n}}-1.$ § 245.

That is, *approximately*,

1. The logarithms of numbers *very near unity* are to each other as the *differences of the numbers from unity.*

Thus, $\quad \log 1.000\,001 = .000\,000\,434\,294$;

$\quad \log 1.000\,002 = .000\,000\,868\,588$; and

$.000\,000\,434\,294 : .000\,000\,868\,588 = .000\,001 : .000\,002.$

Also (§ 238)

$$ly'^{\frac{1}{n}} - ly^{\frac{1}{n}} : ly''^{\frac{1}{n}} - ly^{\frac{1}{n}} = y'^{\frac{1}{n}} - y^{\frac{1}{n}} : y''^{\frac{1}{n}} - y^{\frac{1}{n}}.$$

That is, approximately,

2. *The differences of the logarithms* of numbers *very near unity* are to each other as *the differences of the numbers.*

See example under 1, above; and under § 335. *b, c.*

§ 334. *a.*) Let y be not <2, and $n = 2^{60}$, so that y^n may

be a very large number; also let $D =$ a small number, as 1, 2, &c. Then (§§ 316. 2; 115; 333. 1)

$$l(y^n+D)-l\,y^n = l\left(\frac{y^n+D}{y^n}\right) = l\left(1+\frac{D}{y^n}\right) \doteq \frac{D}{y^n} \doteq D,$$

if y^n is constant (§ 247. 3). That is, approximately,

If *large* numbers differ by quantities *very small in comparison with themselves*, the *differences of their logarithms* will be as the *differences of the numbers.*

Thus, in the common system, if the logarithms are carried to only seven places of decimals, we have

$$l\,10\,000 = 4; \quad l\,10\,001 = 4.000\,043\,4;$$
$$l\,10\,002 = 4.000\,086\,8 \text{ \&c.};$$

where, for equal increments of the number, we have equal increments of the logarithm.

NOTE. We must not extend the series far, because the differences of the numbers would cease to be very small compared with the numbers themselves.

§ 335. *b.*) We have, by (26) and (29),

$$L\,y^{\frac{1}{n}} = y^{\frac{1}{n}}-1, \text{ and } L\,y'^{\frac{1}{n}} = y'^{\frac{1}{n}}-1.$$

$$\therefore \quad L\,y'^{\frac{1}{n}} - L\,y^{\frac{1}{n}} = y'^{\frac{1}{n}} - y^{\frac{1}{n}}.$$

That is, approximately,

The *difference* of the *Naperian* logarithms of two numbers *very near unity* is EQUAL to the *difference* of the numbers.

Thus, Log 1 $= 0$;

Log 1.000 001 $=$ 0.000 000 999 999, or 0.000 001;

Log 1.000 002 $=$ 0.000 001 999 998, or 0.000 002.

That is, the numbers differing by .000 001, their Naperian logarithms differ, within an extremely small fraction, by the same quantity.

c.) In any system whatever, we have

$$\log y^{\frac{1}{n}} = M(y^{\frac{1}{n}}-1); \quad \log y'^{\frac{1}{n}} = M(y'^{\frac{1}{n}}-1).$$

$$\therefore \qquad \log y'^{\frac{1}{n}} - \log y^{\frac{1}{n}} = M(y'^{\frac{1}{n}} - y^{\frac{1}{n}}).$$

That is, in *any* system, approximately,

The difference of the logarithms of two numbers *very near unity* is equal to the *difference of the numbers multiplied by the modulus* of the system.

Thus, in the common system,

$$\log 1 \qquad\qquad = 0;$$
$$\log 1.000\,001 = 0.000\,000\,434\,29;$$
$$\log 1.000\,002 = 0.000\,000\,868\,58.$$

That is, having diff. numbers $= .000\,001$,
we have diff. logarithms $= .000\,001 \times .434\,29$. § 329. (23)

NOTE. If we make $M=1$, the logarithms become Naperian, and this principle becomes identical with the preceding.

d.) Reasoning as in § 334, we shall find, that, if any number whatever receive an increment *very small in comparison with itself,* the corresponding *increment of the logarithm* is approximately equal to *the modulus into the increment of the number, divided by the number.* Thus,

$$l\,(y^n+D)-l\,y^n = l\left(1+\frac{D}{y^n}\right) = M\frac{D}{y^n}.$$ [See §§ 337. *a*; 332. (28), (29)]. Thus (§ 334),

$$\log 10\,001 - \log 10\,000 = .434 \times \tfrac{1}{10000} = .000\,043\,4.$$

So, $\log 9865 = 3.994\,097$, $\log 9866 = 3.994\,141$. And

$$\log 9866 - \log 9865 = .434\,294 \times \tfrac{1}{9865} = .000\,044.$$

§ 336. *e.*) If a number exceed *unity* by *a very small* quantity, its logarithm exceeds *zero* by a very small quantity. Now, §332. (29), $M = \dfrac{\log y^{\frac{1}{n}}}{y^{\frac{1}{n}}-1}.$

Therefore, the *modulus* approximately expresses or measures the ratio (§ 230) of the *infinitesimal* excess of the *logarithm above zero,* to the corresponding *infinitesimal* excess of the *number above unity.*

Thus (§ 335. *c*), $M=\dfrac{.000\,000\,434\,29}{.000\,001}=.434\,29.$

f.) Or (§ 335. *d*), $M=\dfrac{l\,(y^n+D)-l\,y^n}{D\div y^n}.$ That is,

The modulus approximately expresses the ratio of an *infinitesimal increment of any logarithm*, to the corresponding *increment of the corresponding number, divided by the number itself.*

Thus (§ 335. *d*) $M=\dfrac{.000\,044}{\frac{1}{9865}}=.434\,294.$

§ 337. Putting, in (27) of § 332, $1+y$ instead of y, and of course y in place of $y-1$, we have

$$\log\,(1+y)=M(y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+\&c.).\qquad (a)$$

Putting $-y$ for y in (a), we have

$$\log\,(1-y)=M(-y-\tfrac{1}{2}y^2-\tfrac{1}{3}y^3-\tfrac{1}{4}y^4-\&c.).\qquad (b)$$

Subtracting (b) from (a),

$$l\,(1+y)-l\,(1-y)=l\,\frac{1+y}{1-y}=2M\,(y+\tfrac{1}{3}y^3+\&c.).\qquad (c)$$

$$\therefore\ \log\,(1+y)=\log\,(1-y)+2M(y+\tfrac{1}{3}y^3+\tfrac{1}{5}y^5+\&c.).\qquad (31)$$

This series converges with tolerable rapidity, when y is a small fraction. Thus if $y=.1$, we have

$$\log\frac{1+.1}{1-.1}=\log\frac{11}{9}=$$

$$\log 11-\log 9=2M\left(\frac{1}{10}+\frac{1}{3(10)^3}+\frac{1}{5(10)^5}+\&c.\right).$$

$$\therefore\qquad \log 11=\log 9+2M\left(\frac{1}{10}+\frac{1}{3(10)^3}+\frac{1}{5(10)^5}+\&c.\right).$$

Here, if no more than seven places of decimals are required, the fourth term of the series may be neglected.

Now, § 329. (23), $M=0.434\,294\,48$;

and, §§ 316. 3; 322, $\log 9=2\log 3=.954\,242\,5.$

$\therefore \quad \log 11 = .954\,242\,5 + .868\,588\,96(.1 + .000\,333\,3 + \&c.).$

$$\therefore \qquad \log 11 = 1.041\,392\,8.$$

§ 338. Making in (c)

$$y = \frac{1}{2z+1}, \text{ we have } \frac{1+y}{1-y} = \frac{z+1}{z}; \text{ and}$$

$$\log \frac{z+1}{z} = 2M\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right).$$

$$\therefore \log(z+1) = \log z + 2M\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \&c.\right). \quad (32)$$

Thus, if $z = 10$, we have $z+1 = 11$; and

$$\log 11 = \log 10 + .868\,588\,96\left(\frac{1}{21} + \frac{1}{3(21)^3} + \&c.\right).$$

This series may be summed thus;

$$\begin{array}{r|lr}
2z+1 = 21 & .868\,588\,96 \quad = 2M & \\
(2z+1)^2 = 441 & .041\,361\,38 \div 1 = & .041\,361\,38 \\
(2z+1)^2 = 441 & \qquad 93\,79 \div 3 = & 31\,26 \\
 & \qquad\quad 21 \div 5 = & 4 \\
 & & \overline{.041\,392\,68.}
\end{array}$$

$$\therefore \qquad \log 11 = \log 10 + .041\,392\,68 = 1.041\,392\,68.$$

This series converges much more rapidly than the preceding. Many still more rapidly converging series have been devised. We shall give, however, but a single example.

§ 339. Let $z+1 = u^2$. Then $z = u^2 - 1$, and (32) becomes

$$\log u^2 = \log(u^2-1) + 2M\left(\frac{1}{2u^2-1} + \frac{1}{3(2u^2-1)^3} + \&c.\right).$$

Or, as $\qquad \log u^2 = 2\log u,$

and $\qquad \log(u^2-1) = \log(u+1) + \log(u-1),$

we have $\qquad \log(u+1) = 2\log u - \log(u-1) -$

$$2M\left(\frac{1}{2u^2-1} + \frac{1}{3(2u^2-1)^3} + \frac{1}{5(2u^2-1)^5} + \&c.\right). \quad (33)$$

Thus, if $u=12$, we have

$$\log 13 = 2\log 12 - \log 11 - 2M\left(\frac{1}{287}+\frac{1}{3(287)^3}+\&c.\right).$$

Now $2\log 12 = 2(\log 3+\log 4) = 2(\log 3 + 2\log 2)$,

$$= 2(.477\,121\,25 + .602\,060) = 2.158\,364\,;$$

and (§ 336) $\qquad \log 11 = 1.041\,392\,68.$

The series may be summed thus;

$$\begin{array}{r|l}
2u^2-1 = 287 & .868\,588\,96 \quad = 2M \\
(2u^2-1)^2 = 62\,369 & .003\,026\,44 \div 1 = .003\,066\,44 \\
 & \qquad\qquad 5 \div 3 = \qquad\qquad\quad 2 \\
 & \qquad\qquad\qquad\quad .003\,026\,45.
\end{array}$$

$\therefore \qquad \log 13 = 2.158\,362\,5 - 1.041\,392\,68 - .003\,026\,45.$

$$= 1.113\,943.$$

For all larger numbers, the first term of this series will give the value correctly to eight places.

Note. The logarithms of the *prime* numbers only need be computed by such processes; the logarithms of all other numbers being found by the proper combination of the logarithms of primes. Thus,

$$\log 4 = 2\log 2; \quad \log 6 = \log 2 + \log 3; \ \&c.$$

The logarithms of 2 and 3 may be found from formula (32), and the logarithms of 5 and 7 from (32) or (33).

EXPONENTIAL THEOREM.

§ 340. In the equation $a^x = y$, we have found x in terms of y; i. e. a logarithm in terms of the corresponding number.

We shall now find y in terms of x; i. e. a number in terms of its logarithm.

Put $\qquad y[=a^x=(1+a-1)^x] = [(1+a-1)^n]^{\frac{x}{n}};$

n being any number whatever, and of which the value of y is entirely independent (§ 276; 277; 280. c).

Developing,

$$(1+a-1)^n = 1 + n(a-1) + \frac{n(n-1)}{1\,.\,2}(a-1)^2 + \&c.;$$

or, $$(1+a-1)^n = 1+An+Bn^2+Cn^3+\text{\&c.};$$

A, B, C, &c. being functions of a; and, evidently,

$$A = a-1-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\text{\&c.} = La. \qquad \text{§ 326. } b.$$

Then we have

$$y = [(1+a-1)^n]^{\frac{x}{n}} = [1+(An+Bn^2+\text{\&c.})]^{\frac{x}{n}}.$$

$$\therefore\ y = 1+\frac{x}{n}(An+Bn^2+\text{\&c.})+\frac{x(x-n)}{1\,.\,2n^2}(An+Bn^2+\text{\&c.})^2$$

$$+\frac{x(x-n)(x-2n)}{1\,.\,2\,.\,3n^3}(An+Bn^2+\text{\&c.})^3+\text{\&c.}; \qquad \text{§ 295. } i.$$

or $$y = 1+x(A+Bn+\text{\&c.})+\frac{x(x-n)}{1\,.\,2}(A+Bn+\text{\&c.})^2$$

$$+\frac{x(x-n)(x-2n)}{1\,.\,2\,.\,3}(A+Bn+\text{\&c.})^3+\text{\&c.}$$

$$\therefore (\text{§ } 277) \quad y = a^x = 1+Ax+\frac{A^2x^2}{1\,.\,2}+\frac{A^3x^3}{1\,.\,2\,.\,3}+\text{\&c.} \qquad (34)$$

$$\text{Or} \quad a^x = 1+La\,.\,x+\frac{(La)^2x^2}{1\,.\,2}+\frac{(La)^3x^3}{1\,.\,2\,.\,3}+\text{\&c.} \qquad (35)$$

§ 341. *a.*) We know the value of A from §§ 325. (12); 328. *e.* But, if we did not know it, we might find it from the equation (34) itself. Thus,

Let $$x = \frac{1}{A}. \qquad \text{Then } (\text{§ } 330)$$

$$a^{\frac{1}{A}} = 1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+\text{\&c.} = e. \qquad (36)$$

$$\therefore (\text{§ 52. N.}) \qquad a = e^A. \qquad (37)$$

That is, $\frac{1}{A}$ is the logarithm of e to the base a (§ 328. *d*, *f*); and A is the logarithm of a to the base e (i. e. the Naperian logarithm of a [§ 328. *e*]).

b.) Or thus; taking the logarithms of both members of (37),

$$\log a = A \log e. \qquad \text{§ 316. 3.}$$

$$\therefore \qquad A=\frac{\log a}{\log e}=\text{Log } a=\frac{1}{\log e}=\frac{1}{M}, \text{ } a \text{ being the base.}$$

That is, A is the *reciprocal of the modulus* of the system of logarithms whose base is a.

NOTE. The logarithms may be taken in any system, provided both be taken in the same; the logarithms of two numbers having the same ratio in one system as in another (§ 328. b).

§ 342. c.) We have found (§ 341) the value of A, in terms of a. We may, if we prefer, assign a value to A, and find the corresponding value of a.

Thus, let $\qquad A=1.$

Then, $\qquad a^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\&c.$

Making $x=1$, $\qquad a=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.=e.$

§ 343. d.) This is called the *exponential* theorem; and y, in the equation, $y=a^x$, is called an exponential function of x. On the other hand, x is called a *logarithmic* function of y; being the logarithm of y to the base a.

e.) These two classes of functions are also called TRANSCENDENTAL[d] functions; as transcending the elementary operations of Algebra.

EXPONENTIAL EQUATION.

§ 344. The equation, $\qquad a^x=b,$
is called an *exponential equation.* If a is the base of a system of logarithms, we have simply

$$x=\log b.$$

But if a is not the base of a system of logarithms, take the logarithms of both sides of the equation, in any system. The common system is usually most convenient. Then,

$$\log (a^x)=\log b; \qquad \text{or } x \log a=\log b. \qquad \text{§ 316. 3.}$$

(*d*) Lat. transcendo, *to exceed, surpass.*

$$\therefore \qquad x=\frac{\log b}{\log a}.$$

1. Given $12^x=20$, to find x.

$x \log 12=\log 20$; i. e. $1.079\,181\,25x=1.301\,030$.

$$\therefore \qquad x=\frac{\log 20}{\log 12}=\frac{1.301\,030}{1.079\,181\,25}=1.205\,57\text{ \&c.}$$

2. Given $60^x=7$, to find x. *Ans.* $x=.475\,273$.

3. Given $125^x=25$, to find x. *Ans.* $x=\frac{2}{3}$.

§ 345. *d.*) If the equation be of the form,

$$x^x=b,$$

we have $$x \log x=\log b.$$

This equation, can be most conveniently solved by the method of *trial.* For this purpose, find by trial two approximate values of x. Substitute these values successively in the equation,

$$x \log x=\log b,$$

and note the error in each result. Then

Diff. of the results : Diff. of the assumed numbers = the error in either result : the correction to be applied to the corresponding assumed number.

This correction, being applied, will give a nearer approximation to the true value of x. This new value may now be taken as one of the assumed numbers, and a still closer approximation obtained; and so on.

1. Given $x^x=100$, to find x.

Here we have $\quad x \log x=\log 100=2.$

Also, since $\quad 3^3=27$, and $4^4=256$, the value of x lies between 3 and 4.

Substituting 3 and 4 successively, we have

$$3 \log 3=3\times 0.477\,121\,25=1.431\,363\,75\,;$$

whence $\quad 2-1.431\,363\,75=.568\,636\,25$, error;

also $\quad 4 \log 4 = 4 \times 0.602\,059\,99 = 2.408\,239\,96;$

whence $\quad 2 - 2.408\,239\,96 = -.408\,239\,96$, error;

and $\quad 0.976\,876\,21 =$ difference of results.

$\therefore \quad .967\,876 : 1 = -.408\,239\,96 : -.418$, correction.

$\therefore \quad 4 - .418 = 3.582 = x$ nearly.

Again, we find $x > 3.582$, and < 3.6. Therefore, substitute these values, and repeat the operation. Thus,

$$3.582 \log 3.582 = 3.582 \times 0.554\,126\,5 = 1.984\,881.$$

$\therefore \quad 2 - 1.984\,881 = .015\,119$, error.

$$3.6 \log 3.6 = 3.6 \times 0.556\,301\,9 = 2.002\,689.$$

$\therefore \quad 2 - 2.002\,689 = -.002\,689$, error.

Also, $\quad .017\,811 =$ difference of results.

$\therefore \quad 017\,811 : .018 = -.002\,689 : -.002\,717$, correction.

$\therefore \quad 3.6 - 0.002\,717 = 3.597\,283 = x$ nearly.

That is, $\quad 3.597\,283^{3.597283} = 100.$

2. Given $x^x = 5$, to find x. *Ans.* $x = 2.1293$.

3. Given $x^x = 2000$, to find x.

Ans. $x = 4.827\,822\,63$.

4. Given a, m and l of an equimultiple series, to find n (§ 264. *b*).

Ans. $n = \dfrac{\log l - \log a}{\log m} + 1$.

5. Let $a = 2$, $l = 162$, and $m = 3$; and find n.

Ans. $n = 5$.

6. In how many years will p dollars amount to A dollars, at r per cent. compound interest?

We have (§ 258. 5) $A = p(1+r)^t$.

Ans. $t = \dfrac{\log A - \log p}{\log(1+r)}$.

7. In what time will \$100 amount to \$200 (i. e. in what time will a sum of money double itself), at 6 per cent. compound interest? *Ans.* 11.89 years.

Here $p = 100$, $A = 200$, and $1+r = 1.06$.

Notes. (1.) The solution of most questions relating to compound interest may be greatly facilitated by the use of logarithms. (2.) The formulæ of compound interest apply also to the increase of population in a country.

8. Find r from the formula, $A=p(1+r)^t$. See § 258. 5, 9.

$$\textit{Ans.}\ \log(1+r)=\frac{\log A-\log p}{t}.$$

9. The population of the United States in 1830 was 12 866 000, and in 1840, 17 068 000. What was the yearly rate of increase?

Here $A=17\,068\,000$, $p=12\,866\,000$, and $t=10$.

$$\therefore\quad \log(1+r)=\frac{\log 17\,068\,000-\log 12\,866\,000}{10}.$$

$$=\frac{7.232\,183-7.110\,118}{10}=.012\,206\,5.$$

$$\therefore\quad 1+r=1.0285;\ \text{and}\ r=.0285=2\tfrac{17}{20}\ \text{per cent.}$$

10. At the same rate, what will be the population in 1850?

Here $p=17\,068\,000$; $r=.0285$, and $t=10$.

$$\therefore\quad A[=p(1+r)^t]=17\,068\,000\ (1.0285)^{10}.$$

$$\therefore\quad \log A=\log 17\,068\,000+10\log 1.0285.$$

Ans. $A=22\,654\,000$.

11. In how many years will the population amount to 50 000 000? *Ans.* In 38.24 years from 1830.

12. If the number of slaves in the United States in 1830 was 2 009 000, and in 1840, 2 487 000, what was the yearly rate of increase? *Ans.* $2\frac{1}{6}$ per cent.

13. At the same rate, what will be the number in 1850? in 1860? *Ans.* 3 078 700, in 1850; 3 811 000, in 1860.

14. The population of Virginia in 1830 was 1 211 400, and in 1840, 1 239 700; that of New York in 1830 was 1 918 600, and in 1840, 2 428 900. What was the yearly rate of increase in each state?

Ans. In Virginia, .0023, or less than $\frac{1}{4}$ of 1 per cent;
in New York, .0238, or more than $2\frac{1}{3}$ per cent.

CHAPTER XVI.

THEORY OF EQUATIONS.

§ 346. We shall confine ourselves here to the consideration of equations containing but *one* unknown quantity.

1. If the exponents of the unknown quantity in such an equation be all *integral*, or if their *differences* be all *integral, the degree of the equation* is correctly expressed by *the difference between the greatest and the least of those exponents* (§§ 40. *a*; 51. *b*).

2. But, if the difference between any two of the exponents be *fractional*, this difference between the greatest and least, obviously, may not express the degree of the equation. Thus, evidently,

$$x^2 + ax + bx^{\frac{1}{3}} + c = 0$$

is not of the second degree.

Reducing, however, the exponents to a common denominator, we have

$$x^{\frac{6}{3}} + ax^{\frac{3}{3}} + bx^{\frac{1}{3}} + c = 0,$$

which may be said to be of the $\frac{6}{3}$ degree. In fact, if we make $\sqrt[3]{x} = y$, we shall have

$$y^6 + ay^3 + by + c = 0;$$

an equation of the sixth degree in respect to y (i. e. in respect to $x^{\frac{1}{3}}$).

Hence, when the difference between any two of the ex-

ponents is fractional, the degree of the equation is the difference between the greatest and least exponents, *expressed in terms of the least common denominator of all the exponents.* Thus,

$$x^{\frac{1}{2}} + ax^{\frac{1}{3}} + b \; (= x^{\frac{3}{6}} + ax^{\frac{2}{6}} + b) = 0 \text{ is of the } \tfrac{3}{6} \text{ degree.}$$

$$x^{\frac{1}{3}} + ax^{\frac{1}{4}} + bx^{\frac{1}{6}} + c = 0 \text{ is of the } \tfrac{4}{12} \text{ degree.}$$

§ 347. It is evident that equations of this kind can be expressed in integral degrees, by reducing their exponents to a common denominator, m, and substituting a new unknown quantity for the mth root of x (i. e. by putting $x^{\frac{1}{m}} = y$).

Hence, we shall need to consider equations of *integral* degrees only, and shall suppose them reduced to the following form, viz.

$$x^n + A_1x^{n-1} + A_2x^{n-2} \;.\;.\;.\; + A_{n-1}a + A_n = 0. \quad (1)$$

We shall also assume, that every equation has at least one root.

Note. A single symbol, as X, or $f(x)$, is sometimes put for the first member of an equation. Thus, $X = 0$, or $f(x) = 0$.

DIVISIBILITY.—ROOTS.

§ 348. Let a be a root (§ 39) of equation (1). Then,

$$a^n + A_1a^{n-1} + A_2a^{n-2} \;.\;.\;.\;. + A_{n-1}a + A_n = 0.$$

$$\therefore \quad A_n = -a^n - A_1a^{n-1} - A_2a^{n-2} \;.\;.\;.\;. - A_{n-1}a.$$

Substituting this value of A_n in (1), we have

$$(x^n - a^n) + A_1(x^{n-1} - a^{n-1}) \;.\;.\;.\; + A_{n-1}(x - a) = 0.$$

Now this expression is divisible by $x—a$ (§§ 81, 96). Hence (compare § 213. 5),

If a be a root of the equation,

$$x^n + A_1x^{n-1} \;.\;.\;.\;.\;.\;.\;. + A_{n-1}x + A_n = 0,$$

the first member of the equation is divisible by $x—a$.

a.) This principle may be demonstrated otherwise; thus,

If we actually divide the first member of (1) by $x-a$, we shall have, representing the quotient by Q, and the remainder by R,

$$f(x)=x^n+A_1x^{n-1}\ldots+A_n=(x-a)Q+R=0. \quad (2)$$

Now, if a is a root of the equation, the supposition,

$$x=a \text{ (i. e. } x-a=0),$$

reduces the first member of (1) to zero (§§ 39, 211).

$\therefore$ $R=0$, and *the division is perfect* (§ 82. *g*).

Thus, 4, 5 and -1 are roots of the equation,

$$x^3-8x^2+11x+20=0.$$

See if the first member is divisible by $x-4$, $x-5$, and $x-(-1)[=x+1]$.

b.) If a is not a root of the equation, the substitution of a for x will not reduce the first member of (1) to zero. In that case, we shall have, from (2),

$f(a)=a^n+A_1a^{n-1}\ \ldots+A_{n-1}a+A_n=R.$ That is,

If a polynomial, a function of x, of the form,

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ \ldots\ldots+A_n,$$

be divided by x—a, the remainder will be the same function of a, that the given polynomial is of x; i. e. it will be what the given polynomial becomes, when a is substituted for x. See § 211. 1.

NOTES. (1.) The remainder is independent of x. For, if it contained x, the division might be continued further. R, therefore, since it does not contain x, will have the same relation to a, whatever value is given to x. (2.) It is evident also from this principle, that, if a is a root of the equation, the remainder will be zero, and the division perfect.

1. Divide $x^3+A_1x^2+A_2x+A_3$ by $x-a$.
Rem. $a^3+A_1a^2+A_2a+A_3$.

2. Divide $x^3-8x^2+11x+20$ by $x-a$.
Rem. $a^3-8a^2+11a+20$.

§ 349. *c.*) Conversely, *if the first member of* (1) *be divisible by* $x-a$, then

$$R = a^n + A_1 a^{n-1} + A_2 a^{n-2} \ldots\ldots + A_n = 0;$$

i. e. the substitution of a for x satisfies the equation (§ 39); and, therefore, a is a *root of the equation.*

d.) Hence, to determine whether a is a root of the equation,

$$x^n + A_1 x^{n-1} \ldots\ldots + A_n = 0,$$

we have only to divide the first member by $x-a$. And,

(1.) If the division is perfect (§ 82. *g*), a is a root; (2.) if it is not perfect, the remainder is the value of the first member, with a substituted for x (§ 348. *b*).

§ 350. 1. Find whether 3 is a root of the equation,

$$x^5 - x^4 - 25x^3 + 85x^2 - 96x + 36 = 0.$$

Divide by $x-3$; thus (§ 86),

$$\begin{array}{l|l} 1 - 1 - 25 + 85 - 96 + 36 & \;1 \\ \quad + 3 + \;\; 6 - 57 + 84 - 36 & +3 \\ \hline 1 + 2 - 19 + 28 - 12, \quad 0, & \text{the remainder.} \end{array}$$

Hence, the remainder being zero, 3 is a root.

2. Find whether 4 is a root of the same equation.

In performing these divisions, the first coefficient, being 1, need not be written. Thus,

$$\begin{array}{l|l} 1 - 1 - 25 + 85 - \;\; 96 + \;\; 36 & 4 \\ \quad + 4 + 12 - 52 + 132 + 144 & \\ \hline 1 + 3 - 13 + 33 + \;\; 36, + 180, & \text{the remainder.} \end{array}$$

Consequently, 4 is not a root; and the substitution of 4 for x reduces the first member to 180.

3. What does the first member of the equation,

$$x^4 - 7x^3 - 20x^2 + 30x - 48 = 0,$$

become, when 7 is substituted for x? *Ans.* -818.

This may, of course, be determined by the actual substitution of 7 for x. But we arrive at the same result much more conveniently by dividing by $x-7$, as in the preceding examples.

4. What does the first member of the equation,

$$x^2 - 20x + 96 = 0,$$

become, when 7 and 9 are successively substituted for x?

Ans. 5, and —3.

NUMBER OF ROOTS.

§ 351. Let a_1 be a root of the equation,

$$X = x^n + A_1 x^{n-1} + \text{\&c.} = 0. \qquad (1)$$

Then if we divide by $x - a_1$, we shall have, evidently,

$$X = (x - a_1)(x^{n-1} + B_1 x^{n-2} + B_2 x^{n-3} \ldots + B_{n-1}) = 0;$$

an equation which will be satisfied, if either of its factors be equal to zero. Making

$$x^{n-1} + B_1 x^{n-2} + \text{\&c.} = 0,$$

and supposing a_2 to be one of its roots, the primitive equation will take the form (§ 348),

$$X = (x - a_1)(x - a_2)(x^{n-2} + C_1 x^{n-3} \ldots + C_{n-2}) = 0.$$

We may, obviously, proceed in this way, diminishing the degree of the polynomial by unity at each division, till we have taken out n factors of the form $x - a$.

$$\therefore \quad X = x^n + A_1 x^{n-1} \ldots + A_{n-1} x + A_n =$$
$$(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n) = 0. \qquad (2)$$

Now this equation will be satisfied, if any one of its n factors be equal to zero; i. e. if x be equal to any one of the n quantities, $a_1, a_2 \ldots a_n$. Therefore,

1. Every equation of the form,

$$X = x^n + A_1 x^{n-1} \ldots + A_n = 0,$$

can be resolved into n binomial factors, of the form, $x - a$.

(2.) *Every equation has as many roots as there are units in its degree.* See § 213. 1, 2.

Thus (§ 348. *a*), the equation,

$$x^3 - 8x^2 + 11x + 20 = (x - 4)(x - 5)(x + 1) = 0,$$

has the three roots, 4, 5, —1.

§ 352. Suppose b, a quantity different from any of the roots a_1, a_2, a_3, &c., to be a root of equation (2). Then we have

$$(b-a_1)(b-a_2)(b-a_3)\ .\ .\ (b-a_n)=0,$$

an evident absurdity; because, by hypothesis, b being not equal to any of the quantities, a_1, a_2, &c., no one of the factors, $b-a_1$, $b-a_2$, &c. can be equal to zero. Hence,

The number of roots of an equation cannot be greater than the number of units in its degree.

Hence (§§ 351, 352)

§ 353. *The number of roots of an equation is always* EQUAL *to the number of units in its degree.*

a.) These roots may be all real; or part or all of them may be *imaginary* (§ 216).

b.) Again, they are not always different from one another. Any part, or all of them may be *equal* (§ 205).

An equation will, of course, contain *equal roots*, when its first member contains *equal factors.*

Thus, the equation,

$$x^3-3x^2+3x-1=(x-1)(x-1)(x-1)=0,$$

has three roots, each equal to 1.

c.) If we know a part of the roots of an equation, we may find, by dividing by the corresponding factors, the equation of a lower degree, which contains the remaining roots (§ 351).

1. One root of the equation,

$$x^4-9x^3+19x^2+9x-20=0,$$

is 1. Find the equation which contains the remaining roots. *Ans.* $x^3-8x^2+11x+20=0$.

2. Another root of the same equation is 4. Find the equation containing the other two roots.

Ans. $x^2-4x-5=0$.

3. Find the remaining two roots by § 207 or 208.

4. One root of the equation,

$$x^3 - 1 = 0, \text{ i. e. } x^3 = 1,$$

is, obviously, 1. What are the other roots (§ 207)?

Ans. $\frac{1}{2}(-1 + (-3)^{\frac{1}{2}})$, and $\frac{1}{2}(-1 - (-3)^{\frac{1}{2}})$.

d.) Either of the roots of the last equation, being cubed, will produce 1. Thus, every number has *three cube roots*; one, real; and two, imaginary.

In like manner, every number has *four fourth* roots; and, in general, n nth roots.

§ 354. *e.*) The principle of § 353 may be applied to equations of fractional degrees (§ 346. 2).

Thus, the number of the roots of the equation,

$$x^{\frac{3}{2}} - 7x^{\frac{1}{2}} + 6 = 0, \qquad \text{may be said to be } \tfrac{3}{2}.$$

For we find $\qquad x^{\frac{1}{2}} = 1, 2, \text{ or } -3;$

and, consequently, $\qquad x = 1, 4, \text{ or } 9.$

Now these three values of x correspond to *six* values of $x^{\frac{1}{2}}$, only *three* of which satisfy the equation; as will be seen, if we take $x = -1, -2$, or $+3$. The values of x, therefore, i. e. the roots, may properly be said to be *half roots* (§ 12).

So, the equation, $\quad x^{\frac{1}{3}} - 2 = 0$, i. e. $x^{\frac{1}{3}} = 2$,

obviously gives $\qquad x = 8$, or $x - 8 = 0$.

But $x - 8 = (x^{\frac{1}{3}} - 2)(x^{\frac{1}{3}} + 1 - \sqrt{-3})(x^{\frac{1}{3}} + 1 + \sqrt{-3})$, only one of which partial or component factors (§ 12), with the corresponding *partial root*, is found in the given equation. The equation may, therefore, be said to contain only *one third* of a root. See § 221. 2, 3.

COEFFICIENTS.

§ 355. Let $a_1, a_2 \ldots . a_n$ be the roots of an equation. Then we shall have

$$x^n + A_1 x^{n-1} \ldots + A_n = (x-a_1)(x-a_2) \ldots (x-a_n) = 0.$$

Multiplying (§ 283), $x^n + A_1 x^{n-1} \ldots\ldots + A_n =$

$$\begin{array}{r|r|r|l}
x^n - a_1 & x^{n-1} + a_1a_2 & x^{n-2} - a_1a_2a_3 & x^{n-3} \ldots \pm a_1a_2 \ldots a_n. \\
-a_2 & +a_1a_3 & -a_1a_2a_4 & \\
-a_3 & \vdots & \vdots & \\
\vdots & +a_1a_n & -a_1a_2a_n & \\
-a_n & +a_2a_3 & -a_2a_3a_4 & \\
 & \vdots & \&c. & \\
 & +a_2a_n & & \\
 & \&c. & &
\end{array}$$

Hence (§ 277),

$$A_1 = -a_1 - a_2 - a_3 \ldots\ldots - a_n.$$

$$A_2 = a_1a_2 + a_1a_3 \ldots + a_1a_n \ldots + a_2a_n + \&c.$$

$$A_3 = -a_1a_2a_3 - a_1a_2a_4 \ldots\ldots - a_1a_3a_n \ldots\ldots - \&c.$$

$$A_4 = a_1a_2a_3a_4 + a_1a_2a_3a_5 + \&c.$$

$$A_n = \pm\, a_1a_2a_3a_4a_5 \ldots a_n.$$

That is,

(1.) *The coefficient of the second term* is equal to *the sum of the roots with their signs changed* (§ 213. 3).

(2.) The coefficient of the *third* term is equal to the sum of the *products* of the roots taken *two and two* (§ 213. 4); (3.) that of the *fourth* term, to the sum of their products taken *three and three;* and so on, the *signs* of the roots being *changed* in every case.

(4.) The *absolute* term (i. e. the coefficient of x^0 [§ 208]) is the product of the roots taken *all together*, with their signs changed.

a.) It is evident, that, in the third, fifth, seventh, &c. terms, the number of factors being even, the result will be the same, whether the signs of the roots be changed or not (§ 213. 4).

b.) The last term will be *positive* or *negative*, according as the number of *positive* roots is *even* or *odd* (§ 215. 1, 2).

c.) If the roots be all *negative*, the factors will be of the

form $x+a_1$, $x+a_2$, &c., and the terms will, evidently, all be *positive* (§ 215. 1, 3); if the roots be all *positive*, the terms will be *alternately positive and negative* (§ 215. 1, 3).

d.) If the coefficient of the *second* term be *zero*, the sum of the *positive roots* is numerically equal to the sum of the *negative* roots (§ 214. 1).

e.) *Every* root of the equation is a *divisor of the last term;* and, hence, if the last term be *zero*, one of the roots must be *zero* (§ 214. 2); or rather, in this case, the equation becomes of the $(n-1)$th degree (§ 203).

1. Form the equation, whose roots are 2, 3, and —4.
Ans. $(x-2)(x-3)(x+4)=x^3-x^2-14x+24=0$.

2. Form the equation, whose roots are 1, 1, 2, and 3.
Ans. $x^4-7x^3+17x^2-17+6=0$.

3. Given the roots, 2, $-1+\sqrt{-3}$, $-1-\sqrt{-3}$; to find the equation. *Ans.* $x^3-8=0$.

FORM OF THE ROOTS.

§ 356. Let the equation, $x^n+A_1x^{n-1}\ldots+A_n=0$, have its coefficients all integral (the coefficient of the first term being unity); it is required to determine whether it can have a fractional root.

If possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a root of the equation. Then we shall have

$$\frac{a^n}{b^n}+A_1\frac{a^{n-1}}{b^{n-1}}\ldots\ldots+A_n=0.$$

Multiplying by b^{n-1}, and transposing,

$$\frac{a^n}{b}=-A_1a^{n-1}-A_2a^{n-2}b\;\ldots-A_nb^{n-1}.$$

Now all the terms of the *second member* of this equation, are whole numbers, while the first member is an irreducible fraction. That is, we have an irreducible fraction equal to a whole number; which, evidently, is impossible. Hence,

If the coefficient of the first term be unity, and the other coefficients all integral, the equation cannot have a fractional root.

a.) It is not, therefore, to be inferred, that all the roots are integral. They may be either *integral*, *irrational* (§§ 153, 175), or *imaginary* (§ 23. *f.* 2).

§ 357. Let the coefficients of the equation, $X=0$, be all real; and let $a+b\sqrt{-1}$ be a root of the equation.

The quantity $b\sqrt{-1}$ can have resulted only from the extraction of an even root, which must have given, at the same time, $-b\sqrt{-1}$ (§ 23. *f.* 1). Consequently, $a-b\sqrt{-1}$ must be a root of the equation.

Otherwise; the sum and product of the roots (§ 355. 1, 4) must both be real. Therefore, if one root be $a+b\sqrt{-1}$, another must be $a-b\sqrt{-1}$, so that their product (§ 186) and sum may both be free from imaginary expressions. Hence,

If the coefficients of an equation be all real, the number of its imaginary roots must be even (§ 217. I.).

a.) Thus, there may either be *no* (§ 63. N.) imaginary roots, or there may be *two, four*, &c. Hence,

b.) Cor. I. *Every* equation of an *odd* degree has at least *one real root,* with a *sign* (see *c.* below,) *different from that of the last term* (i. e. of the coefficient of x^0).

c.) We have [§§ 186, 162]

$$(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2+b^2,$$

a positive quantity (§ 11. N.). Hence,

Cor. II. If *all* the roots of an equation are *imaginary, the last term* must be *positive* (§ 216). Hence,

Cor. III. *Every* equation of an *even* degree, whose *last term is negative,* has at least *two real* roots; *one positive, and the other negative* (§§ 68. *a*; 215. 2).

1. Given the roots, 5, $3+\sqrt{-4}$, $3-\sqrt{-4}$; to form the equation. *Ans.* $x^3-11x^2+43x-65=0$.

2. Form the equation, whose roots are $-6+5\sqrt{-1}$, $-6-5\sqrt{-1}$, $1+\sqrt{-4}$, and $1-\sqrt{-4}$.

Ans. $x^4+10x^3-42x^2-62x+305=0$.

3. Form the equation, whose roots are $2, -2, 1+\sqrt{-3}$, and $1-\sqrt{-3}$. *Ans.* $x^4-2x^3+8x-16=0$.

§ 358. *d.*) Again (§ 218. *h*),

$$(x-a-b\sqrt{-1})(x-a+b\sqrt{-1})=(x-a)^2+b^2;$$

a result necessarily positive for every *real* value of x. Consequently,

Cor. IV. (1.) The product of *all* the *imaginary* factors is *positive for every real* value of x. Hence,

(2.) The *sign* of the *first member*, for any *real* value of x, depends on the *real* factors. And,

(3.) If *all* the roots are *imaginary*, the first member will be *positive* for *every real* value of x.

e.) The product,

$$(x-a-\sqrt{-b})(x-a+\sqrt{-b})=x^2-2ax+a^2+b^2,$$

of the factors corresponding to each *pair* of imaginary roots, or *conjugate*[c] roots, as they are sometimes called, is real. Hence,

Cor. V. Every equation may be resolved into *real* factors; of the *first* degree, corresponding to the *real* roots; and of the *second* degree, corresponding to each *pair* of *imaginary* roots.

SIGNS OF THE ROOTS.

§ 359. Let a be a root of the equation,

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ \ldots\ +A_{n-1}x+A_n=0. \quad (1)$$

Changing the signs of the alternate terms, we have

$$x^n-A_1x^{n-1}+A_2x^{n-2}-A_3x^{n-3}+\&c.=0; \quad (2)$$

or (§ 44. *a*), changing all the signs of (2),

$$-x^n+A_1x^{n-1}-A_2x^{n-2}+A_3x^{n-3}-\&c.=0. \quad (3)$$

(c) Lat. conjugo, *to join together.*

The equations (2) and (3) are, obviously, the same; as will be seen by transposing, in each, all the negative terms to the other side.

Now, if $+a$ be substituted for x in (1), and $-a$, in (2) when n is an *even* number, or in (3) when n is *odd*, the results will be precisely alike. But the substitution of $+a$ in (1) reduces the first member to 0. Consequently, the substitution of $-a$ in (2) or (3) reduces the first member to 0, and therefore $-a$ is a root of the equations (2) and (3). Hence,

If the signs of the alternate terms in an equation be changed, the signs of all the roots will be changed.

Form the equations, whose roots are 1, 2, and 3; and -1, -2, and -3.

§ 360. A *permanence*[f] of signs occurs when *two successive* terms are affected each by the *same* sign; a *variation*, when their signs are *different*. Thus, $x+a=0$ exhibits a *permanence*, and $x-a=0$, a *variation;* the first corresponding to a *negative*, and the second, to a *positive* root.

I. Let the signs of the terms in their order, in any *complete* equation be $+ + - - - + -$, and let a new factor, $x-a=0$, corresponding to a new *positive* root, be introduced. The signs will be as follows, viz.

$$\begin{array}{llllllll}
+ & + & - & - & - & + & - & \\
+ & - & & & & & & \\
\hline
+ & + & - & - & - & + & - & \\
 & - & - & + & + & + & - & + \\
\hline
+ & \pm & - & \mp & \mp & + & - & +.
\end{array}$$

Now, in this result, it is manifest, that each permanence is changed into an ambiguity; and that, whether there be one, or any greater number, of double signs, the single signs immediately preceding and following are always unlike. Hence the number of permanences may be diminished, but cannot be increased.

Hence, the number of signs being one greater than be-

(*f*) Lat. permaneo, *to continue.*

fore, the number of variations also must be at least one greater.

Now the equation, $x-a=0$, containing one positive root, has one variation. Consequently, as every additional positive root introduces, at least, one additional variation,

The number of VARIATIONS *can never be less than the number of* POSITIVE *roots.*

II. (1.) By like reasoning it can be shown, that the introduction of a negative root (i. e. of the factor $x+a$) will introduce at least one permanence; and that, therefore,

The number of PERMANENCES *cannot be less than the number of* NEGATIVE *roots.*

(2.) Or, if we change the signs of the alternate terms, the variations will evidently become permanences, and the permanences, variations; and the negative roots will, at the same time, become positive (§ 359).

But the number of *variations* in this equation cannot be less than the number of its *positive* roots. Therefore, the number of *permanences* in the primitive equation cannot be *less* than the number of its *negative* roots.

Hence, universally, in a complete equation,

§ 361. *The number of* POSITIVE *roots cannot be greater than the number of* VARIATIONS *of sign; nor the number of* NEGATIVE *roots, greater than the number of* PERMANENCES.

NOTE. A *complete* equation of the nth degree,

$$x^n+A_1x^{n-1} \; . \; . \; . \; . \; . +A_{n-1}x+A_n=0,$$

must, obviously, contain $n+1$ consecutive powers of x; and, of course, $n+1$ terms (§§ 195, 196).

1. How many permanences and variations in the equation, whose roots are 2, 2, and -5?

Ans. The equation is

$$(x-2)(x-2)(x+5)=x^3+x^2-16x+20=0;$$

showing *one* permanence, and *two* variations, as we have seen there must be.

2. How many permanences and variations in the equation, whose roots are 1, 2, 4, and -4?

a.) The whole number of variations and permanences must, evidently, be equal to the degree of the equation (the equation being complete, or, if not complete, being rendered so by the introduction of cyphers, as in § 362).

Therefore,

Cor. I. If the roots of an equation be *all real*, the number of *positive* roots must be *equal* to the number of *variations;* and the number of *negative* roots, to the number of *permanences.* See § 218. 1, 2, 3.

§ 362. *b.*) If any term of the equation be wanting, a cypher may be put in its place; and, obviously, either sign may be given to it without affecting the roots of the equation.

Thus, the equation,

$$x^2 + 25 = 0,$$

may be written $\qquad x^2 \pm 0 + 25 = 0.$

Now, in this equation, if the upper sign be taken with the middle term, there will be no variations; and, of course, the equation has no positive root. But, if the lower sign be taken, there will be no permanences; and, therefore, the equation has no negative root. Consequently, the roots of the equation are imaginary (§ 353).

So, in the equation,

$$x^3 \pm 0 + 4x + 7 = 0,$$

if the upper sign be taken with the second term, there will be no variation, and no positive root; and, if the lower sign be taken, there will be but one permanence, and, of course, not more than one negative root. The other two roots are, therefore, imaginary.

The equations, $\qquad x^2 \pm 0 - 25 = 0,$

and $\qquad x^3 \pm 0 - 4x + 7 = 0,$

exhibit the same number of permanences and variations, whether we take the upper or lower sign before the mis-

sing term; and, consequently, it cannot be inferred that the roots are not all real.

Hence,

§ 363. Cor. II. If the introduction of $+0$ in place of a missing term gives *a different number of permanences and variations from that* given by the introduction of -0, the equation contains *imaginary* roots.

c.) This will, obviously, happen, if the *terms immediately preceding and following the deficient term* have *like* signs.

d.) Also, if *two or more successive* terms be wanting, then, supplying the terms, the first of the supplied terms may always have the same sign as the term following all the deficient terms. Consequently, the equation *must have imaginary roots.*

Thus, in the equation,

$$x^3 - 0 \pm 0 - 1 = 0,$$

if we take the upper sign before the third term, we have three variations, to which negative roots cannot correspond; if we take the lower sign, we have two permanences, to which positive roots cannot correspond. Two of the roots, then, can be neither positive nor negative; and must, of course, be imaginary.

§ 364. *e.*) It is evident also, that, the greater the number of deficient terms, the greater difference can be made between the numbers of variations and of permanences, respectively; and, therefore, the greater will be the number of imaginary roots of which we shall be assured. Thus, it is easily seen, that,

(1.) If an *odd* number $(2m+1)$ of *consecutive* terms be wanting, the number of imaginary roots must be at least $2m+2$, if the signs of the terms immediately preceding and following the deficient terms be *like;* and at least $2m$, if they be *unlike.*

a.) Thus, in the equation,

$$x^4 \pm 0 \pm 0 \pm 0 + 1 = 0,$$

we find, if we take the upper signs throughout, no variations; and, if we take, alternately, the lower and upper signs, no permanences. Hence, there must be $4(=2m+2)$ imaginary roots.

b.) In the equation,

$$x^4-0-0\pm 0-1=0,$$

we may have one permanence and three variations, or one variation and three permanences. Hence, we may have one positive and one negative root; and must have $2(=2m)$ imaginary roots.

(2.) Also, the deficiency of an *even* number $(2m)$ of consecutive terms indicates at least $2m$ imaginary roots.

c.) Examine the permanences and variations of the equation,

$$x^3-0\pm 0-1=0.$$

NOTES. (1.) Giving to the first cypher in the last example, and to the first two in the last but one, the sign of the term following them all, we have an odd number $(2m-1)$ of terms wanting, preceded and followed by terms of like signs. Wherefore, by 1, above, there must be at least $2m(=2m-1+1)$ imaginary roots.

(2.) It should be remembered, that there may be more imaginary roots than are thus *indicated;* and that there are frequently imaginary roots when no terms are wanting (§§ 216; 218. 4).

TRANSFORMATION.

§ 365. Let it be required to transform the equation

$$X=x^n+A_1x^{n-1}+A_2x^{n-2}\ \ldots+A_{n-1}x+A_n=0, \quad (1)$$

into another whose roots shall be less than those of the given equation by x'.

The roots of the new equation will, of course, be equal to $x-x'$. Let $y=x-x'$. Then $y+x'=x$; and, if we substitute $y+x'$ for x, we shall have a polynomial of the same value as before, but expressed in terms of $y(=x-x')$ instead of x. Thus,

$$X=(y+x')^n+A_1(y+x')^{n-1}+A_2(y+x')^{n-2}\ \ldots\ldots$$
$$\ldots\ldots+A_{n-2}(y+x')^2+A_{n-1}(y+x')+A_n=0.$$

Developing,

$$\left.\begin{array}{l|l|l} y^n + nx' & y^{n-1} \ldots\ldots + nx'^{n-1} & y + \quad x'^n \\ \quad + A_1 & \quad + (n-1)A_1x'^{n-2} & + A_1x'^{n-1} \\ & \quad + (n-2)A_2x'^{n-3} & + A_2x'^{n-2} \\ & \qquad : \qquad : & \qquad : \\ & \qquad + 2A_{n-2}x' & + A_{n-2}x'^2 \\ & \qquad\quad + A_{n-1} & + A_{n-1}x' \\ & & \qquad + A_n \end{array}\right\} = 0. \quad (2)$$

Or, putting B_1, B_2, &c., for the coefficients of y^{n-1}, y^{n-2}, &c.

$$X = y^n + B_1y^{n-1} + B_2y^{n-2} \ldots + B_{n-1}y + B_n = 0. \quad (3)$$

Or, again,

$$X = (x-x')^n + B_1(x-x')^{n-1} \ldots + B_{n-1}(x-x') + B_n = 0; \quad (4)$$

where $x-x'$ may be regarded as the unknown quantity.

Now, obviously, the roots of (2), (3) and (4) are the values of $y(= x - x')$; and are, therefore, less by x' than the roots of the given equation (i. e. the values of x).

Hence, the transformation required is effected by the substitution of $y + x'$ (i. e. of $[x - x'] + x'$) for x in the given equation. Thus,

Find an equation, whose roots shall be less by 2 than those of the equation,

$$x^2 - 9x + 20 = 0.$$

Substitute $y + 2$ for x.

Then $\qquad (y+2)^2 - 9(y+2) + 20 = 0,$

or $$\left.\begin{array}{r|l} y^2 + 4 & y + \quad 2^2 \\ -9 & \quad -9\times 2 \\ & \quad +20 \end{array}\right\} = 0, \quad \text{or } y^2 - 5y + 6 = 0,$$

is the equation required, whose roots will be found to be less by 2 than those of the given equation.

§ 366. *a.*) The labor of effecting this substitution may be greatly abridged, especially in the higher equations.

For B_n, i. e. the coefficient of y^0 in the transformed equation (2), is simply what the first member of the given equation becomes, when x' is substituted for x. That is,

$$B_n = f(x').$$

B_{n-1} is formed by *multiplying each term of* B_n *by the exponent of* x' *in that term, and diminishing the exponent by unity.*

B_{n-2} is formed by *multiplying* each term of B_{n-1} *by its exponent of* x', *diminishing the exponent by unity, and dividing* by 2; and so on.

b.) In other words, each term of B_{n-1} is the *first derived function* (§ 292. N. 3) of the corresponding term of B_n; i. e. of $f(x')$.

Each term of B_{n-2} is *half* the first derivative of the corresponding term of B_{n-1}; i. e. half the *second* (§ 292. N. 4) derivative of the corresponding term of B_n.

So, each term of B_{n-3} is *one third* of the first derivative of the corresponding term of B_{n-2}; i. e. *one sixth* of the *third* (§ 292. N. 4) derivative of the term of B_n.

c.) Hence, B_{n-1} is called the first *derived polynomial* of B_n, or of the given equation; and may be expressed by B_n', or by $f'(x')$.

B_{n-2} is half the *second* derived polynomial of the equation, and may be expressed by $\frac{B_n''}{2}$, or by $\frac{f''(x')}{2}$.

$$\text{So, } B_{n-3} = \frac{B_n'''}{2\,.\,3} = \frac{f'''(x')}{2\,.\,3}; \quad B_{n-4} = \frac{B_n^{IV}}{2\,.\,3\,.\,4} = \frac{f^{IV}(x')}{2\,.\,3\,.\,4};$$

$$B_{n-5} = \frac{B_n^{V}}{2\,.\,3\,.\,4\,.\,5} = \frac{f^{V}(x')}{2\,.\,3\,.\,4\,.\,5}; \text{ \&c.}$$

1. Diminish by 2 the roots of the equation,

$$x^2 + 5x + 6 = 0.$$

The transformed equation will be of the form,

$$y^2 + B_1 y + B_2 = 0.$$

And we shall have

$$B_2 = f(x') = x'^2 + 5x' + 6 = 2^2 + 5\times 2 + 6 = 20\,;$$

$$B_1 = B_2' = f'(x') = 2x' + 5 = 2\times 2 + 5 = 9\,;$$

$B_0 = \frac{1}{2}B_2'' = \frac{1}{2}\times 2 = 1$ (B_0 denoting the coefficient of y^2).

$\therefore\quad y^2 + 9y + 20 = 0$ is the equation required.

§ 367. *d.*) A still more convenient method of finding these coefficients results from the form of equation (4).

For, comparing (4) and (1), we have

$$(x-x')^n + B_1(x-x')^{n-1} \;.\;.\;.\;.\;.\; + B_{n-1}(x-x') + B_n$$
$$= x^n + A_1x^{n-1} \;.\;.\;.\;.\;.\; + A_{n-1}x + A_n. \qquad (5)$$

Now every term of the first member of this equation is divisible by $x - x'$, except the last term, B_n; which will be the remainder.

In like manner, every term of the quotient which results from this division is, evidently, divisible by $x - x'$, except the last, B_{n-1}, which will be the remainder; and so on.

But the second member being *absolutely* (§ 37. *d*) equal to the first, its successive divisions by $x - x'$ must result in the same quotients and remainders as the division of the first member.

Hence,

If we divide the given equation by $x - x'$, the remainder will be B_n, *the coefficient of* y^0 in the transformed equation.

If we divide the *resulting quotient* by $x - x'$, the *remainder* will be B_{n-1}, the *coefficient* of y^1; and so on, each of the coefficients being formed by the successive division of the several quotients by $x - x'$.

e.) It is evident also from § 348. *b*, that the first remainder will be B_n [$= f(x')$]; i. e. what X becomes, when x' is substituted for x (§ 350. 2, 3, 4).

1. Transform the equation, $x^2 + 9x + 20 = 0$, into another whose roots shall be less by 5 than those of the given equation.

$$
\begin{array}{r|l|l}
x^2 + 9x + 20 & x - 5 & \\
x^2 - 5x & x + 14 & x - 5 \\
\hline
14x & x + \ 5 & 1 \\
14x - 70 & \quad 19 = B_{n-1}. & \\
\hline
90 = B_n. & &
\end{array}
\qquad \text{Or (§ 86),} \qquad
\begin{array}{r|l}
1 + 9 + 20 & 1 \\
+ 5 + 70 & 5 \\
\hline
1 + 14, + 90, B_n. & \\
+ \ 5 & \\
\hline
1, + 19, B_{n-1}. &
\end{array}
$$

$\therefore$ $y^2 + 19y + 90 = 0$ is the equation required.

2. Find an equation, whose roots shall be less by 3 than those of the equation,

$$x^3 + 10x^2 - 15x + 30 = 0.$$

Neither the first coefficient of the divisor (§ 350. 2), which is always 1, nor the first coefficients of the quotients, each of which is the same as the first coefficient of the dividend, need be written. Thus,

$$
\begin{array}{l}
1 + 10 - 15 + \ 30 \ (3 \\
\quad + \ 3 + 39 + \ 72 \\
\hline
\quad + 13 + 24, + 102 = B_n = B_3. \\
\quad + \ 3 + 48 \\
\hline
\quad + 16, + 72 = B_{n-1} = B_2. \\
\qquad \ 3 \\
\hline
\quad + 19 = B_{n-2} = B_1.
\end{array}
$$

$\therefore$ $y^3 + 19y^2 + 72y + 102 = 0$ is the equation required.

3. Find an equation, whose roots shall be less by -2 (i. e. *greater* by 2), than those of the equation,

$$x^3 + 8x^2 - 20x + 25 = 0.$$

We must here, of course divide by $x - (-2)$; i. e. by $x + 2$. *Ans.* $y^3 + 2y^2 - 40y + 89 = 0.$

4. Find the equation whose roots are less by 1 than those of the equation,

$$x^3 - 2x^2 + 3x - 4 = 0.$$

Ans. $y^3 + y^2 + 2y - 2 = 0.$

5. Find the equation, whose roots are less by 2 than those of the equation,

$$x^5 + 2x^3 - 6x^2 - 10x + 3 = 0.$$

Ans. $y^5 + 10y^4 + 42y^3 + 86y^2 + 70y + 12 = 0.$

6. Diminish by 2.8 the roots of the equation,

$$x^4-12x^2+12x-3=0.$$

We may here either diminish the roots of the equation by 2, and then the roots of that equation by .8, or we may diminish the roots of the given equation at once by 2.8. The former method is generally the more convenient.

Thus,

$$\begin{array}{rrrrrl}
1 & +0 & -12 & +12 & -3 & (2 \\
 & 2 & +4 & -16 & -8 & \\
\cline{1-5}
 & 2 & -8 & -4, & -11 & \\
 & 2 & +8 & 0 & & \\
\cline{2-4}
 & 4 & 0, & -4 & & \\
 & 2 & +12 & & & \\
\cline{2-3}
 & 6, & +12 & & & \\
 & 2 & & & & \\
\cline{2-2}
 & 8 & +12 & -4 & -11 & (.8 \\
 & .8 & +7.04 & +15.232 & +8.9856 & \\
\cline{2-5}
 & 8.8 & +19.04 & +11.232, & -2.0144 & \\
 & .8 & +7.68 & +21.376 & & \\
\cline{2-4}
 & 9.6 & +26.72, & +32.608 & & \\
 & .8 & +8.32 & & & \\
\cline{2-3}
 & 10.4, & +35.04 & & & \\
 & 8 & & & & \\
\cline{2-2}
 & 11.2 & & & &
\end{array}$$

Diminishing the roots by 2, we find the equation,

$$y^4+8y^3+12y^2-5y-11=0.$$

Diminishing the roots of this equation by .8, we have

$$y^4+11.2y^3+35.04y^2+32.608y-2.0144=0\,;$$

the equation required.

7. Diminish by 1.3 the roots of the equation,

$$x^3-7x+7=0.$$

Ans. $x^3+3.9x^2-1.93x+.097=0.$

8. Diminish by 14 the roots of the equation,

$$x^3-17x^2+54x-350=0.$$

Ans. $x^3+25x^2+166x-182=0.$

§ 368. If the coefficient of any power of y in equation (2) of § 365 reduce to zero, that term will be wanting in the new equation. Thus the *second* term will disappear from the equation, if $nx' + A = 0$; i. e. if $x' = -\frac{A}{n}$. Hence,

To make the second term disappear, we must make $x' = -\frac{A}{n}$; i. e. we must *diminish the roots by* $-\frac{A}{n}$; or, which is the same thing, *increase them by* $+\frac{A}{n}$.

a.) This will be evident otherwise; thus,

The sum of the n roots of the primitive equation is $-A$ (§ 355. 1). Now if each of the roots be increased by $\frac{A}{n}$, their sum will be increased by A; and will, of course, be equal to $-A + A = 0$.

1. Remove the second term from the equation,

$$x^4 - 4x^3 - 19x^2 + 106x - 120.$$

Here we have $n = 4$, and $A = -4$.

$$\therefore \qquad x' = -\frac{A}{n} = -\frac{-4}{4} = 1;$$

and we must diminish the roots of the equation by 1.

$$\begin{array}{l}
1 - 4 - 19 + 106 - 120 \;(1 \\
\underline{\quad 1 - 3 - 22 + 84} \\
-3 - 22 + 84, - 36 = B_4. \\
\underline{\quad 1 - 2 - 24} \\
-2 - 24, + 60 = B_3. \\
\underline{\quad 1 - 1} \\
-1, - 25 = B_2. \\
\underline{\quad 1} \\
\quad 0 = B_1.
\end{array}$$

$\therefore \quad y^4 - 25y^2 + 60y - 36 = 0$ is the equation required.

Transform the following equations in like manner.

2. $x^3 - 3x^2 - 4x + 12 = 0$.

Ans. $x^3 - 7x + 6 = 0$.

3. $x^5+14x^4+12x^3-20x^2+14x-25=0.$

Ans. $y^5-78y^3+412y^2-757y+401=0.$

4. $x^2+2px+q^2=0.$ *Ans.* $y^2+(q^2-p^2)=0.$

b.) The last result leads to the common solution of the equation. For, by transposition,

$$y^2=p^2-q^2; \text{ and } \therefore y=\pm(p^2-q^2)^{\frac{1}{2}}.$$

But
$$y=x+p.$$

$$\therefore \qquad x+p=\pm(p^2-q^2)^{\frac{1}{2}}.$$

$$\therefore \qquad x=-p\pm(p^2-q^2)^{\frac{1}{2}}.$$

c.) If we would remove any other term from the equation, we must make the coefficient of that term in (2) of § 365 equal to zero, and find the corresponding values of x'. By the substitution of a value so found for x', that term will, of course, vanish.

It is obvious, that, to remove the *third* term, we must solve an equation of the *second* degree; for the *fourth*, one of the *third* degree, and so on.

To remove the *last* term, we must solve an equation of the nth degree; in fact, the given equation itself, with x' substituted for x. The values of x' found from this equation will, therefore, evidently be the roots of the given equation.

§ 369. If, in the general equation,

$$x^n+A_1x^{n-1}+A_2x^{n-2} \;.\;.\;.\;.\; +A_{n-1}x+A_n=0,$$

we put $y=mx$ (i. e. substitute $\frac{y}{m}$ for x), we shall have

$$\frac{y^n}{m^n}+A_1\frac{y^{n-1}}{m^{n-1}} \;.\;.\;.\;.\;.\;.\;.\; +A_{n-1}\frac{y}{m}+A_n=0;$$

or (§ 46) $y^n+A_1my^{n-1} \;.\;.\; +A_{n-1}m^{n-1}y+A_nm^n=0;$ an equation whose roots are m times those of the primitive equation. Hence,

An equation will be transformed into another, *whose roots shall be equal to the roots of the first multiplied by any*

number, as m, if we *multiply the second term* of the given equation by m, *the third* by m^2, and so on. Hence,

Cor. I. An equation having *fractional coefficients* may be changed into another with *integral coefficients*, by transforming it so that its roots shall be those of the given equation *multiplied by a common multiple of the denominators.*

Cor. II. If the *coefficients* of the second, third, &c. terms of an equation be respectively *divisible* by m, m^2, &c., then the *roots* of the equation are of the form mx, and consequently m is a *common measure* of them.

1. Transform the equation,

$$3x^3 + 4x^2 - 5x + 6 = 0,$$

into another whose roots shall be three times those of the given equation.

Here $m = 3$. $\therefore y = 3x$, and $x = \frac{1}{3}y$.

Ans. $3y^3 + 12y^2 - 45y + 162 = 0$;

or, $y^3 + 4y^2 - 15y + 54 = 0.$

2. Transform the equation,

$$x^3 + \tfrac{2}{3}x^2 + \tfrac{3}{4}x - \tfrac{5}{2} = 0,$$

into an equation with integral coefficients.

Ans. $x^3 + 8x^2 + 108x - 4320 = 0.$

§ 370. If in the general equation,

$$x^n + A_1x^{n-1} + A_2x^{n-2} \;.\;.\;.\;. + A_{n-1}x + A_n = 0,$$

we substitute $\frac{1}{y}$ for x, we shall have

$$\frac{1}{y^n} + A_1\frac{1}{y^{n-1}} + A_2\frac{1}{y^{n-2}} \;.\;.\;.\;.\;. + A_{n-1}\frac{1}{y} + A_n = 0;$$

or, clearing of fractions, and reversing the order of the terms,

$$A_ny^n + A_{n-1}y^{n-1} \;.\;.\;.\;. + A_2y^2 + A_1y + 1 = 0;$$

an equation, whose roots are the *reciprocals* of the roots of the given equation. Hence,

To transform an equation into another, *whose roots shall*

be the reciprocals of the roots of the first, we have only to *reverse the order of the coefficients.*

a.) Cor. We may also, evidently, transform an equation into another, whose roots shall be *greater or less than the reciprocals* of the roots of the given equation, or *multiples* of those reciprocals, by applying the processes of §§ 367, 369 to the coefficients taken in a reverse order.

b.) It may happen, that the coefficients, when taken in the reverse order, shall be the same as when taken directly. In such a case, the transformed will obviously be identical with the given equation; and will have the same roots. Consequently, as the roots of the transformed are the reciprocals of those of the given equation, and, at the same time, are identical with them, *one half of the roots of the given equation must be the reciprocals of the other half.*

Thus the roots will be $a, \frac{1}{a}$; $b, \frac{1}{b}$; &c.

c.) If the coefficients of corresponding terms are *numerically* equal, but have *unlike signs*, the same is true of the roots, in *every* equation of an *odd* degree; and, in every one of an *even* degree, *whose middle term is wanting.* For, in both these cases, if all the signs of the transformed equation be changed, (which will not affect the value of the roots,) the transformed will be identical with the primitive equation.

§ 371. *d.*) Such equations (§ 370. *b, c*), which remain the same, when $\frac{1}{x}$ is substituted for x, are called *recurring*[g] or *reciprocal* equations.

e.) The general form of a recurring equation is, obviously,

$$x^n + A_1x^{n-1} + A_2x^{n-2} \ldots\ldots + A_2x^2 + A_1x + 1 = 0.$$

Recurring equations have certain peculiar properties, which will be considered hereafter.

(g) Lat. recurro, *to run back.*

LIMITS OF THE ROOTS.

§ 372. In the equation,

$$(x-a_1)(x-a_2)(x-a_3)\ \ldots = 0,$$

let a_1, a_2, a_3, &c. be the *real* roots, taken in the order of their magnitude; i. e. $a_1 > a_2$, $a_2 > a_3$, &c.

If now b_1, $> a_1$, be substituted for x, we have

$$(b_1-a_1)(b_1-a_2)(b_1-a_3)\ \ldots\ (b_1-a_n),\ \textit{positive};$$

all the real factors being *positive* (§§ 68. *a*; 358. 1, 2).

If b_2, $< a_1$ and $> a_2$, be substituted for x, we have

$$(b_2-a_1)(b_2-a_2)(b_2-a_3)\ \ldots\ (b_2-a_n),\ \textit{negative};$$

one of the real factors being *negative* (§ 68. *a*).

So, if we substitute b_3, $< a_2$ and $> a_3$, the product will be *positive; two* of the real factors being *negative*, and the rest, *positive*.

In like manner, the substitution of b_4, $< a_3$ and $> a_4$, will give a *negative;* of b_5, $< a_4$ and $> a_5$, a *positive* result; and so on. Hence,

(1.) If a quantity, *greater than the greatest real root* of an equation, be substituted for x, the result will be *positive;* and,

(2.) If quantities *intermediate* between the roots, beginning with the greatest, be successively substituted for x, the results will be *alternately negative and positive.*

The roots of the equation,

$$x^3 - 5x^2 + 2x + 8 = 0,$$

are 4, 2, and -1. Substitute 5, 3, 1, 0, and -2 for x, and observe the signs of the results.

§ 373. *a.*) Hence,

Cor. I. When two quantities are successively substituted for x, if the results have *like* signs, there is an *even;* if *unlike* signs, an *odd* number of real roots between those quantities.

NOTE. The even number may be 0 (§ 63. N.).

b.) From 1, and Cor. I., it is evident, that,

Cor. II. If a number *less than the least* real root be substituted for x, the result will be *positive or negative*, according as the number of real roots is *even or odd.*

c.) If the degree of the equation be *odd*, the substitution of $+\infty$ for x will render the first member *positive;* and of $-\infty$, *negative.* Hence (§ 373. *a*),

Cor. III. (1.) *Every* equation of an *odd* degree must have at least *one* real root (§ 357. *b*); and (2.) the *whole number* of its real roots must be *odd.*

d.) If the degree be *even*, and the *last term negative*, the substitution either of $+\infty$ or of $-\infty$ will render the first member *positive;* and the substitution of 0 will render it *negative.* Hence,

Cor. IV. (1.) *Every* equation of an *even* degree has an *even* (§ 373. N.) number of real roots; and (2.) every equation of an *even* degree, whose *last term is negative*, has at least *two* real roots, one *positive* and the other *negative* (§ 357. Cor. III.).

§ 374. *e.*) If the substitution of p, and of every number greater than p, renders the result *positive*, then p is *greater than the greatest* real root; and is called a *superior limit* of the roots.

f.) So, if, the *signs of the alternate terms being changed* (§ 359), the substitution of q, and of every number greater than q, renders the result *positive*, then $-q$ is *less than the least* real root (i. e. it is an *inferior* limit).

§ 375. Let A_h be the *first*, and A_m, numerically the *greatest, negative* coefficient of any *complete* (§ 361. *a*) equation,

$$x^n + A_1 x^{n-1} \;.\;.\; - A_h x^{n-h} \;.\;.\; - A_m x^{n-m} \;.\;.\; + A_n = 0$$

Now, if all the coefficients after A_h be negative, the sum of those terms will be numerically equal to the sum of the preceding, positive terms.

Consequently, any value of x, which renders the sum of the preceding positive terms numerically greater than the sum of the negative terms, is a *superior limit* of the roots.

And, with still greater reason, any value of x, which renders x^n numerically greater than the sum of the negative terms, is a superior limit.

The most unfavorable case possible is, evidently, when *all* the coefficients after A_h are negative, and each of them is equal to A_m, the greatest.

Any value of x, then, which makes

$$x^n > A_m(x^{n-h} + x^{n-h-1} \ldots + x + 1), \tag{1}$$

or (§ 261)

$$x^n > A_m\left(\frac{x^{n-h+1} - 1}{x-1}\right), \tag{2}$$

is a superior limit.

Now (2) will certainly be true, if we have

$$x^n > A_m \frac{x^{n-h+1}}{x-1}; \quad \text{or } 1 > A_m \frac{x^{-\{h-1\}}}{x-1};$$

or

$$x^{h-1}(x-1) > A_m. \tag{3}$$

But $\quad x - 1 < x$, and $(x-1)^{h-1} < x^{h-1}$.

Therefore (3) will be true, if we have

$$(x-1)^{h-1}(x-1)[=(x-1)^h] = A_m; \tag{4}$$

and, with still greater reason, if $(x-1)^h > A_m$. (5)

Also, (4) and (5) give $x - 1 =$, or $> (A_m)^{\frac{1}{h}}$;

or

$$x = \text{, or} > (A_m)^{\frac{1}{h}} + 1. \tag{6}$$

That is, in a complete equation,

§ 376. *If we increase by unity that root of the greatest negative coefficient, whose number is equal to the number of terms preceding the first negative term,* the result will be a *superior limit* of the roots.

Find superior limits of the roots of the following equations.

1. $x^4 - 5x^3 + 37x^2 - 3x + 39 = 0$.

Here $A_m = 5$, and $h = 1$.

$$\therefore \qquad (A_m)^{\frac{1}{h}} + 1 = 5^{\frac{1}{1}} + 1 = 6, \text{ the limit required.}$$

a.) If the *second* coefficient be negative, the limit found will be the *greatest negative coefficient increased by unity.*

2. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$

Ans. $(49)^{\frac{1}{2}} + 1 = 8.$

3. $x^4 + 11x^2 - 25x - 67 = 0.$

Ans. $(67)^{\frac{1}{3}} + 1 = 6.$

b.) If the signs of the alternate terms be changed (§ 359), and a superior limit be found, that limit with its sign changed will be an *inferior* limit; or, as it is sometimes called, a *superior* limit of the *negative* roots.

c.) A number, which is *numerically less* than the least positive or negative root is sometimes called an *inferior* limit of the *positive*, or of the *negative* roots.

Let the equation be found, whose roots are the reciprocals of the roots of the given equation; and let the superior limits of the positive and negative roots of this new equation be found.

Now those roots of the new equation, which are numerically the greatest, are the reciprocals of those of the given equation, which are numerically the least.

Therefore, the *reciprocals* of the *superior* limits of the positive and negative roots of the new equation will be the *inferior* limits of the positive and negative roots of the given equation.

LIMITING, OR SEPARATING EQUATION.

§ 377. Let a_1, a_2, a_3, &c. be the real roots, taken in the order of their magnitude, of the equation,

$$x^n + A_1 x^{n-1} \ . \ . \ . + A_{n-1} x + A_n = 0. \qquad (1)$$

Diminishing the roots of this equation by x', we have (§ 365)

$$y^n + B_1 y^{n-1} + B_2 y^{n-2} \ldots + B_{n-1} y + B_n = 0; \quad (2)$$

in which (§§ 365; 366. *c*)

$$B_{n-1} = f'(x') = nx'^{n-1} + (n-1)A_1 x'^{n-2} \ldots + A_{n-1}. \quad (3)$$

Also, B_{n-1}, is the sum of the products of the roots, with their signs changed, of equation (2) (i. e. of the products of $x'-a_1$, $x'-a_2$, .. $x'-a_n$), taken $n-1$ at a time (§ 355). That is,

$$\left.\begin{aligned} B_{n-1} = {} & (x'-a_2)(x'-a_3)(x'-a_4) \ldots (x'-a_n) + \\ & (x'-a_1)(x'-a_3)(x'-a_4) \ldots (x'-a_n) + \\ & (x'-a_1)(x'-a_2)(x'-a_4) \ldots (x'-a_n) + \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & (x'-a_1)(x'-a_2)(x'-a_3) \ldots (x'-a_{n-1}); \end{aligned}\right\} (4)$$

each term consisting of $n-1$ factors; and, of course, each factor being found in every term but one.

If now, in this value of B_{n-1}, we make $x' = a_1, a_2, a_3$, &c., successively, we shall have (§ 68. *a*)

$$B_{n-1} = (a_1-a_2)(a_1-a_3)(a_1-a_4) \ldots = +.+.+ \ldots = +;$$
$$B_{n-1} = (a_2-a_1)(a_2-a_3)(a_2-a_4) \ldots = -.+.+ \ldots = -;$$
$$B_{n-1} = (a_3-a_1)(a_3-a_2)(a_3-a_4) \ldots = -.-.+ \ldots = +;$$
&c.

That is, if we substitute a_1, a_2, a_3, &c. for x' in B_{n-1}, the results are *alternately positive and negative.*

Hence (§ 372), the real roots of $B_{n-1} = 0$ lie between a_1, a_2, a_3, &c.; and therefore, putting x in place of x', we have the equation, $B_{n-1} = nx^{n-1} + (n-1)A_1 x^{n-2} + (n-2)A_2 x^{n-3} \ldots + A_{n-1} = 0$; whose *real roots severally lie between those of the given equation;* and which is thence called the *separating* or *limiting* equation.

a.) B_{n-1}, we have seen (§ 366), is the first derived polynomial of the given equation. That is,

$$f'(x) = 0, \text{ or } X' = 0$$

is the limiting equation of

$$f(x) = 0, \text{ or } X = 0.$$

Hence, the separating or limiting equation is properly called the *derived* equation.

b.) It is obvious, that if $f(x) = 0$ have real roots (as assumed in the investigation), the *greatest* and *least* are respectively *greater* and *less* than the *greatest* and *least* real roots of $f'(x) = 0$.

EQUAL ROOTS.

§ 378. *c.*) If the given equation have *two roots equal*, as $a_2 = a_1$, the factor $x - a_1$ will, evidently, be found in *each* of the terms of B_{n-1} [§ 377. (4)]; and, consequently, when $x = a_1$, we shall have B_{n-1} [$= f'(x)$] $= 0$; i. e. a_1 will be a root of $f'(x) = 0$.

So, if $a_3 = a_2 = a_1$, the factor $(x - a_1)^2$ will be found in each of the terms of B_{n-1}, i. e. of $f'(x)$; and the equation, $f'(x) = 0$, will have *two* roots equal to a_1; and so on.

d.) On the other hand also, it is evident, that no factor can exist in *all* the terms of B_{n-1}[$= f'(x)$], unless it enter more than once in $f(x)$, i. e. in the given equation; and, that, if a factor appear any number of times in $f'(x)$, it must be contained *once oftener* in $f(x)$.

e.) Hence, to find whether an equation has equal roots,

Form the *derived* or *limiting* equation, $f'(x) = 0$; and find the *greatest common divisor* (§ 104), D, of the polynomials, $f(x)$ and $f'(x)$.

Make $D = 0$, and find its roots. Each of these roots will be contained *once oftener in the primitive equation*, $f(x) = 0$, than in $D = 0$.

f.) If $f(x)$ and $f'(x)$ have *no common divisor*, the given equation, $f(x) = 0$, has, of course, *no* equal roots.

1. Given $f(x) = x^3 - 4x^2 + 5x - 2 = 0$, to find whether it has equal roots.

The derived equation is

$$f'(x) = 3x^2 - 8x + 5 = 0;$$

and the greatest common divisor of $f(x)$ and $f'(x)$ is

$$x-1.$$

Hence, there are two roots equal to 1.

Dividing $f(x)$ by $(x-1)^2$ (§ 86),

$$\begin{array}{r|l} 1 & 1-4+5-2 \\ +2 & \quad +2-4 \\ -1 & \qquad\; -1+2 \\ \hline & 1-2, \end{array}$$

we find, for the remaining factor, $x-2=0$.

Therefore, the roots of $f(x)=0$ are 1, 1, and 2.

2. Given $x^4-6x^3+13x^2-12x+4=0$, to find the equal roots, if there be any. *Ans.* 1, 1, 2, and 2.

3. Given $x^3+5x^2+3x-9=0$, to find the equal roots. *Ans.* -3, and -3.

NOTE. Between two equal roots of an equation, there can evidently be no intermediate root, unless it be equal to each of them.

Thus, the derivative of the equation,

$$x^2+2px+q^2=0, \quad \text{is } 2x+2p=0, \text{ or } x+p=0;$$

and the root of this derived equation is $-p$, which lies between the two roots of the given equation,

$$-p+\surd(p^2-q^2), \quad \text{and } -p-\surd(p^2-q^2).$$

Now, if q becomes *nearly* equal to p, the quantity under the radical becomes *small*, and the two roots become *nearly* equal. Also, if q becomes *equal* to p, the radical disappears; and the roots become *equal*, taking the form $x=-p\pm 0$.

Thus, the separating root is always intermediate between the *unequal* roots; and is the *limit* to which they approach, as they become equal.

STURM'S THEOREM.

§ 379. STURM'S THEOREM is a method, discovered by M. Sturm in 1829, of finding the *exact number*, and, nearly, the *situation*, of the *real roots* of an equation.

The number of the *real* roots being known, the number of *imaginary* roots is known of course (§ 353, *a*).

Let $X = x^n + A_1 x^{n-1} \; . \; . \; . \; . + A_{n-1} x + A_n = 0$ be an equation which contains *no equal* roots; and let X' represent its first derived polynomial (§§ 366. *c*; 292. N. 3).

Let also the process of finding the greatest common divisor (§ 104) be applied to the polynomials, X and X', as follows; viz.

Divide X by X', till a remainder is obtained of a degree lower than X'.

Change all the signs of this remainder, and represent the resulting quantity by X_2.

Divide X' by X_2, change the signs of the remainder, and designate the result by X_3.

Continue this process, *always changing the signs of the remainders,* till a remainder is obtained independent of x.

NOTES. (1.) This last remainder will *not* be *zero;* because, by hypothesis, the equation does not contain equal roots; and, therefore, the polynomials X and X' have no common measure (§§ 104; 378. *f*).

(2.) In performing these divisions, any *positive* factor not found in one of the polynomials may be introduced or rejected, in the other (§ 100. *a*).

(3.) The numbers, $_2$, $_3$, &c., are used to distinguish the functions, X_2, X_3, &c., from simple derived functions, which would be more appropriately denoted by X'', X''', &c. (§ 292. N. 4).

§ 380. The result of the above operations, representing the successive quotients by Q_1, Q_2, &c., may be expressed as follows; viz.

$$\left.\begin{aligned} X &= X' Q_1 - X_2 ; \\ X' &= X_2 Q_2 - X_3 ; \\ X_2 &= X_3 Q_3 - X_4 ; \\ \vdots \quad & \quad \vdots \quad \vdots \quad \vdots \\ X_{n-2} &= X_{n-1} Q_{n-1} - X_n ; \end{aligned}\right\} \qquad (1)$$

X_n representing the final remainder, which is *independent* of x, and, as we have seen, *not equal to zero.*

§ 381. I. Now, obviously, any one of these functions may become equal to zero for particular values of x. We

must inquire, whether *two consecutive* functions can become zero *at the same time;* i. e. for *the same value* of x.

Suppose that X_2 and X_3 become, at the same time, equal to zero. Then, making, in the third of equations (1), $X_2 = 0$, and $X_3 = 0$, we find $X_4 = 0$.

So, from $X_3 = 0$, and $X_4 = 0$, we find $X_5 = 0$; and so on, till, from the last equation, we find $X_n = 0$, which is contrary to the hypothesis.

Or, proceeding in like manner in the other direction, if X_2 and X_3, or any other two consecutive functions become zero simultaneously, there must also result at the same time, $X = 0$ and $X' = 0$. This, again, is impossible, because the roots of $X' = 0$ are intermediate between those of $X = 0$ (§ 377. *a*); and moreover, there are no equal roots (§ 378. *c*).

Hence,

No two consecutive functions of the series, X, X', X_2, &c., *can become zero at the same time;* i. e. for the same value of x.

§ 382. II. Again, let any one of the functions, as X_3 become zero for a particular value of x.

Making $X_3 = 0$ in the equation,

$$X_2 = X_3 Q_3 - X_4,$$

we have $$X_2 = -X_4.$$ Hence,

If a particular value of x reduces one of the functions to zero, *the adjacent functions must have unlike signs for that value of x.*

§ 383. Let now different values, as p, q, &c., be substituted for x in the functions, X, X', X_2, X_3, &c.; and let the resulting signs of the several functions be written in order, and the number of their variations be noted.

And, in the first place, the signs of the functions will remain unchanged, and the number of their variations, of course, unaffected, so long as q is less than the least (§ 373. *b*) real root of the equations,

$$X = 0,\ X' = 0,\ X_2 = 0,\ X_3 = 0, \text{ \&c.}$$

But, if q becomes equal to a root of one of the equations, the corresponding function will become zero; and, as q increases still more, the function will appear with its sign changed (§ 373. *a*).

We must inquire, what will be the effect of this change of sign on the order of the signs, and on the number of their variations.

§ 384. 1. First, let q be the smallest of all the roots of the equations (1), and let it be a root of one of the auxiliary equations, as $X_3 = 0$.

Then we shall have $X_3 = 0$, and $X_2 = -X_4$. That is, X_2 and X_4 will have unlike signs (§ 382). Moreover, neither of them can become zero at the same time with X_3 (§ 381).

We know also, that neither X_2 nor X_4 can have changed its sign; because we have not passed any of the roots of $X_2 = 0$, or $X_4 = 0$, q being the least of all the roots.

Therefore, whatever may have been the sign of X_3, before it became zero, the signs of X_2 and X_4 having been unlike, the three signs must have exhibited one variation and one permanence. Thus, they must have been either

$$+ \pm -, \text{ or } - \pm +.$$

If now we substitute for x a quantity greater than the least root of $X_3 = 0$, and less than the least root of $X_2 = 0$ and $X_4 = 0$, the signs of X_2 and X_4 will remain as they were; while the sign of X_3 will be changed (§ 373. *a*).

The signs will then stand thus, viz.

$$+ \mp -, \text{ or } - \mp +;$$

still showing one variation and one permanence, as before.

The same reasoning, obviously, applies to any function intermediate between X and X_n. Hence,

The substitution of a *root of an intermediate* equation, or a *change of sign* of one of the *intermediate* functions does *not* affect the *number of variations* of sign in the series.

§ 385. 2. We need, therefore, to consider the case only, in which X changes its sign in consequence of passing a root of the primitive equation, $X=0$.

In examining this case, we must remember, that, if $X=0$ have real roots, the least of them is less than the least real root of $X'=0$ (§ 377. *b*); also, that X being one degree higher than X', one of the equations, $X=0$ and $X'=0$, must have an *odd*, and the other, an *even* number of real roots (§ 373. *c*, *d*).

Consequently, when we substitute for x a quantity less than the least real root of $X=0$, X and X' must have unlike signs (§ 373. *b*).

But X changes its sign in passing the least real root of $X=0$. If, therefore, we substitute for x a quantity greater than that least root of $X=0$, and less than the least root of $X'=0$, X and X' will have *like* signs.

That is, these signs, which before exhibited a *variation*, will now exhibit a *permanence*.

Therefore, as the number of variations in the other functions has undergone no change (§ 384),

The whole *number of variations is diminished by one, in passing a real root of* $X=0$.

a.) The same reasoning will apply to the next real root of $X=0$; and so on.

For suppose, that we have passed any *equal* number of the real roots of $X=0$ and $X'=0$.

Now, if we substitute for x a quantity less than the next greater root of $X=0$, the signs of X and X' will be *unlike* (§ 373. *b*), and will constitute a *variation*.

But, if we substitute for x a quantity greater than that next root of $X=0$, and less than the succeeding root of $X'=0$, X will change its sign; and the signs of X and X', becoming *like*, will constitute a *permanence*.

b.) In fact, after we pass the least root of $X=0$, X and X' have like signs, till we pass the least root of $X'=0$; when they become unlike, without however producing an

additional variation (§ 384). Then, in passing the next root of $X=0$, the change of sign in X introduces a permanence instead of a variation (§ 385).

Hence,

§ 386. If two quantities, p and q be successively substituted for x in the functions, X, X', X_2, &c.,

The *difference between the number of variations*, produced in the signs of these functions, by the substitution of p and of q, is *always equal to the number of real roots* of the equation $X=0$, *included between those quantities;* i. e. between p and q.

a.) When $X'=0$, X and X_2 have unlike signs (§ 382).

But when $X'=0$, X is alternately *positive and negative.* Therefore X_2 is alternately *negative and positive.*

This principle, which, of course, supposes, that $X=0$ has real roots, will enable us better to understand, how the series of signs loses a variation in passing each real root of $X=0$.

b.) (1.) We may find simply the *whole number* of real roots, by substituting $-\infty$ and $+\infty$ for x in the several functions. In this case, each function will have the sign of its first term.

(2.) Moreover, if we substitute 0 for x, the number of variations lost from $-\infty$ to 0 will give the number of *negative* roots; from 0 to $+\infty$, the number of *positive* roots.

It is obvious also, that the substitution of 0 for x will reduce each function to its last term, which is independent of x.

§ 387. The theorem has been demonstrated on the hypothesis, that the equation contains *no equal* roots (§ 379).

If, however, we have an equation containing equal roots, we shall find a common divisor of X and X'; and a remainder, of course, equal to zero.

If now we divide the functions, X, X', &c., by this great-

est common divisor, we shall obtain a new series of functions, Y, Y', Y_2, &c. Now, it is evident, (1.) that $Y=0$ will contain *no equal* roots; and (2.) that the variations of sign in the series of new functions will be the same as in the primitive series.

For, if the common divisor be positive, the signs will not be affected by the division (§§ 62. a; 80. b); and, if it be negative, all the signs will be changed.

Hence, the theorem is applicable to equations having equal roots.

§ 388. 1. How many real roots has the equation,

$$X = x^3 - 7x + 6 = 0?$$

Here
$$X' = 3x^2 - 7;$$
$$X_2 = 7x - 9;$$
$$X_3 = +.$$

	X	X'	X_2	X_3	
$x = -\infty$ gives	$-$	$+$	$-$	$+$,	3 variations.
$x = 0$ "	$+$	$-$	$-$	$+$,	2 variations.
$x = +\infty$ "	$+$	$+$	$+$	$+$,	0 variation.

Hence, there are *three* real roots; *one negative*, and *two positive.*

We shall find, more nearly, the values of the roots by substituting different numbers for x. Thus,

$x = -4$ gives	$-$	$+$	$-$	$+$,	3 variations;
$x = -3$ "	0	$+$	$-$	$+$;	
$x = -2$ "	$+$	$+$	$-$	$+$,	2 variations;
$x = -1$ "	$+$	$-$	$-$	$+$,	2 variations;
$x = 1$ "	0	$-$	$-$	$+$;	
$x = 1.6$ "	$-$	$+$	$+$	$+$,	1 variation;
$x = 2$ "	0	$+$	$+$	$+$;	
$x = 3$ "	$+$	$+$	$+$	$+$,	0 variation.

We have here found the three roots, -3, 1, and 2 (the

values of x which reduce X to zero). We find also that a variation is lost in passing each of the roots.

2. Find the number and situation of the real roots of the equation,
$$X = x^2 + x - 1 = 0.$$
Here
$$X' = 2x + 1;$$
$$X_2 = +5.$$

$\therefore$	$x = -\infty$ gives	$+ \; - \; +$,	2 variations;
	$x = 0$ "	$- \; + \; +$,	1 variation;
	$x = +\infty$ "	$+ \; + \; +$,	0 variation.

There are, therefore, two real roots; one positive, and the other negative. Moreover,

$x = -2$ gives	$+ \; - \; +$,	2 variations;
$x = -1$ "	$- \; - \; +$,	1 variation;
$x = +1$ "	$+ \; + \; +$,	0 variation.

There is, then, one root between -2 and -1; and one between 0 and 1.

The first figure of the negative root is -1; and, by substituting .1, .2, .3, .4, .5, .6, and .7, we find the first figure of the positive root to be .6.

3. How many real roots has the equation,
$$X = x^3 + 11x^2 - 102x + 181 = 0?$$
Here
$$X' = 3x^2 + 22x - 102;$$
$$X_2 = 122x - 393;$$
$$X_3 = +.$$

Hence, we find *three* real roots; one negative, and two positive situated between 3 and 4.

Now, diminishing the roots (§ 367) of the equations, $X = 0$, $X' = 0$, &c., by 3, we find
$$Y = y^3 + 20y^2 - 9y + 1;$$
$$Y' = 3y^2 + 40y - 9;$$
$$Y_2 = 122y - 27;$$
$$Y_3 = +.$$

These functions show that the two positive roots of $Y=0$ lie between .2 and .3. Consequently, the two positive roots of $X=0$ are between 3.2 and 3.3.

Again, diminishing the roots of $Y=0$, $Y'=0$, &c. by .2, we find

$$Z=z^3+20.6z^2-.88z+.008;$$
$$Z'=3z^2+41.2z-.88;$$
$$Z_2=122z-2.6;$$
$$Z_3=+.$$

Hence, the initial figures of the two positive roots of $Z=0$ are .01 and .02. Consequently, the first three figures of the positive roots of $X=0$ are 3.21 and 3.22.

Also, the sum of the roots (§ 355. 1) is -11.

$\therefore \quad -11-3.21-3.22=-17.4$, the negative root.

4. How many real roots has the equation,

$$X=x^5+2x^4+3x^3+4x^2+5x-20=0\,?$$

Here
$$X'=5x^4+8x^3+9x^2+8x+5;$$
$$X_2=-7x^3-21x^2-42x+255;$$
$$X_3=-13x+14;$$
$$X_4=-.$$

$x=-\infty$ gives $\quad - \quad + \quad + \quad + \quad -$, 2 variations.

$x=+\infty$ " $\quad + \quad + \quad - \quad - \quad -$, 1 variation.

Hence the equation has one real, and four imaginary roots. The real root is, of course, positive (§ 357. *b*); and is found to be between 1 and 2.

a.) When we arrive at a function, as X_m, such that the roots of $X_m=0$ are all imaginary, we need not continue the divisions.

For this function having the same sign for all values of x (§ 358. 1), can never conform to the signs of those beyond it; and no changes of sign in those functions can affect the number of variations in the series (§ 384).

The coefficients of an equation of the *second* degree, show at once, whether its roots are imaginary (§ 216).

In respect to equations of higher degrees, the question is not so easy of solution. It can, however, be determined by applying Sturm's theorem, as to an independent equation.

The roots of $X' = 0$, in the last example, are all imaginary; and X and X' give the same result as the whole series of functions.

NOTE. $X' = 0$ is a *recurring* equation (§ 371), and can be easily solved by a process which will be explained hereafter.

5. How many real roots has the equation,

$$X = x^3 + px + q = 0?$$

Here
$$X' = 3x^2 + p;$$
$$X_2 = -2px - 3q;$$
$$X_3 = -4p^3 - 27q^2.$$

b.) First, let p be *positive.*

Then $-2p$ will be negative; and X_2 will be positive for $x = -\infty$; negative, for $x = +\infty$.

Also, $-4p^3$ will be negative; and, as $-27q^2$ is necessarily negative, X_3 will be negative. Thus,

$x = -\infty$ gives $- \; + \; + \; -$, *two* variations;
$x = +\infty$ " $+ \; + \; - \; -$, *one* variation.

Hence, if p be *positive,* the equation has one real, and two imaginary roots.

c.) Again, let p be *negative.*

Then $-2p$ will be positive; and X_2 will be negative for $x = -\infty$; positive, for $x = +\infty$.

Also, $-4p^3$ will be positive; and when

$$-4p^3 > 27q^2, \text{ i. e. when } -4p^3 - 27q^2 > 0,$$

or (§ 146. *d*) $4p^3 + 27q^2 < 0$, X_3 will be positive.

If these conditions be fulfilled, we shall have,

for $x = -\infty$, $- \; + \; - \; +$, *three* variations;
" $x = +\infty$, $+ \; + \; + \; +$, *no* variation.

Hence, if both p, and $4p^3 + 27q^2$ be *negative,* the equation has *three* real roots.

d.) Again, suppose X_3 to be positive; i. e.

$$4p^3 + 27q^2 < 0.$$

Then $$4p^3 < -27q^2. \qquad \S\, 144. \text{ N.}$$

$\therefore$ (§ 147) $\dfrac{p^3}{27} < -\dfrac{q^2}{4}$, or $\left(\dfrac{p}{3}\right)^3 < -\left(\dfrac{q}{2}\right)^2 < 0. \quad \therefore p < 0.$

For $-\left(\dfrac{q}{2}\right)^2$ is negative. Consequently, $\left(\dfrac{p}{3}\right)^3$ is negative; which cannot be unless p is *negative*.

Hence, if $4p^3 + 27q^2 < 0$, the roots are *all real*.

6. How many real roots has the equation,

$$X = x^2 + px + q = 0?$$

Here $$X' = 2x + p;$$
$$X_2 = p^2 - 4q.$$

First, if X_2 be *positive*,

$x = -\infty$ gives $+ \ - \ +$, *two* variations;
$x = +\infty$ " $+ \ + \ +$, *no* variation;

showing *two* real roots.

Again, if X_2 be *negative*,

$x = -\infty$ gives $+ \ - \ -$, *one* variation;
$x = +\infty$ " $+ \ + \ -$, *one* variation;

showing *no* real root.

Consequently, the roots are *real*, or *imaginary*, according as $p^2 - 4q$ is *positive* or *negative*.

Moreover, when X_2 is *negative* (i. e. $p^2 - 4q < 0$), we have $p^2 < 4q$; or $\frac{1}{4}p^2 < q$; or $(\frac{1}{2}p)^2 < q$; which can happen, only when q is *positive*.

Hence, the roots are real, unless $q > (\frac{1}{2}p)^2 > 0$ (§ 216).

NUMERICAL EQUATIONS.—I. INTEGRAL ROOTS.

§ 389. Let a be an *integral* root of the equation,

$$X = x^n + A_1x^{n-1} + A_2x^{n-2} \;.\;.\; + A_{n-1}x + A_n = 0, \quad (1)$$

the coefficients being all integral.

Then $$a^n + A_1a^{n-1} \;.\;.\; + A_{n-2}a^2 + A_{n-1}a + A_n = 0. \quad (2)$$

Transposing, and dividing by a, we have

$$\frac{A_n}{a} = -a^{n-1} - A_1a^{n-2} \; \ldots\ldots - A_{n-2}a - A_{n-1}, \quad (3)$$

a whole number.

Hence, $A_n \div a$ is a whole number; and a is an *integral* factor, or divisor (§ 80. d) of A_n.

Consequently, all the integral roots of an equation will be found among the divisors of the last term. They will also, of course, be contained between the superior and inferior limits (§ 374) of the roots.

Therefore, we shall find *all the integral* roots of an equation, by the method of §§ 349. d, 350., if we substitute for a, successively, the several factors of the last term, which are included between the limits of the roots.

1. Find the integral roots of the equation,

$$x^4 - 7x^3 + 17x^2 - 17x + 6 = 0.$$

Here, the limits, found by § 374, are 18 and -5. It is evident, however, that there can be no *integral* root greater than 6.

Hence, the only numbers to be tried are 6, 3, 2, 1, -1, -2, and -3.

$$\begin{array}{l}
1 - 7 + 17 - 17 + 6 \quad (1 \\
\underline{\quad + 1 - \ \ 6 + 11 - 6} \\
1 - 6 + 11 - \ \ 6 \quad (2 \\
\underline{\quad + 2 - \ \ 8 + \ \ 6} \\
1 - 4 + \ \ 3 \quad (3 \\
\underline{\quad + 3 - \ \ 3} \\
1 - 1 \quad (1 \\
\underline{\quad + 1} \\
1.
\end{array}$$

Consequently, the roots are 1, 1, 2, and 3 (§ 355. e. 2).

2. Find the integral roots of the equation,

$$X = x^3 + x^2 - 17x + 15 = 0.$$

If X is divisible by $x - a$, it is, evidently, divisible by $a - x$; the signs merely of the quotient being different.

Therefore, arranging the coefficients according to the ascending powers of x (§ 33), and dividing by $3-x$, we have

$$\begin{array}{rrrrrrrr|rr}
 & 15 & - & 17 & + & 1 & + & 1 & & 3 \\
 & & + & 5 & - & 4 & - & 1 & + & 1 \\
\hline
 & & - & 12 & - & 3 & & & & \\
\hline
 & 5 & - & 4 & - & 1 & & & & 1 \\
 & & + & 5 & + & 1 & & & + & 1 \\
\hline
 & 5 & + & 1 & & & & & - & 5 \\
 & & & 1 & & & & & & 1 \\
\hline
 & & & 0 & & & & & & \\
\hline
-1. & & & & & & & & &
\end{array}$$

Hence, the roots are 3, 1, and -5.

a.) In this process, the root, evidently, must divide the first term of each remainder; i. e. the sum of each term of the quotient and the succeeding coefficient.

b.) In fact, transposing A_{n-1} in (3), representing $\dfrac{A_n}{a} + A_{n-1}$ by B, and dividing again by a, we have

$$\frac{B}{a} = -a^{n-2} - A_1 a^{n-3} \;.\;.\;.\;.\;.\; - A_{n-3}a - A_{n-2},$$

a whole number.

In like manner, continuing to transpose the coefficient of a^0, and divide by a, each quotient will be a whole number; and the last quotient will be the coefficient of x^n with its sign changed.

3. Find the integral roots of the equation,

$$x^4 - 27x^2 + 14x + 120 = 0.$$

Here $+7$ and -7 are limits. Moreover, only *two* of the roots can be negative, and *two*, positive (§ 361). Hence, having found two positive roots, we need try no more positive divisors.

$$\begin{array}{r|rrrrrrr}
6 & 120 & + & 14 & - & 27 & \quad 0 & + 1 \\
+1 & & + & 20 & & & & \\
\cline{2-4}
 & & + & 34. & & & & \\
\cline{2-4}
\end{array}$$

$34 \div 6$ not being a whole number, 6 is not a root.

The roots are 4, 3, -2, and -5.

c.) If the equation is not in the common form (i. e. with integral coefficients, the first being unity), it should be reduced (§ 369), and the method applied to the reduced equation.

4. Find the roots of the equation,

$$3x^3 - x^2 - 3x + 9 = 0.$$

Having found the values of y from the transformed equation, we shall have $x = \frac{1}{3}y$.

II. INCOMMENSURABLE ROOTS.

§ 390. Find the roots of the equation,

$$X = x^2 + 5x - 5 = 0.$$

Applying Sturm's theorem, we find

$$X = x^2 + 5x - 5;$$
$$X' = 2x + 5;$$
$$X_2 = +.$$

Hence there is a positive root between .8 and .9, and a negative root, between -5 and -6.

If, now, we diminish (§ 367) the roots of $X = 0$ by .8, one root of the transformed equation,

$$Y = y^2 + 6.6y - .36 = 0,$$

will be between 0 and .1.

Applying Sturm's theorem again, we find

$$Y = y^2 + 6.6y - .36 = 0;$$
$$Y' = y + 3.3;$$
$$Y_2 = +.$$

Hence, there is a root between .05 and .06; and, consequently, the root of $X = 0$ is between .85 and .86.

Again, diminishing the roots of $Y = 0$ by .05, one root of the transformed equation,

$$Z = z^2 + 6.7z - .0275 = 0,$$

will be between 0 and .01; and will be found by the theorem to be between .004 and .005.

Hence, the root of $X=0$ is between .854 and .855.

Note. We might, in the same way, find any number of figures of the root. But the process would be tedious. The nature of the roots, however, of the equations, $Y=0$ and $Z=0$, will suggest a more convenient method of determining the successive figures, as appears in the following sections.

§ 391. We know that the root of the equation,

$$y^2+6.6y-.36=0, \tag{1}$$

is less than .1; i. e. we have $y<.1$, and, of course, $y^2<.01$.

Hence it is evident, that the equation,

$$6.6y+.36=0, \tag{2}$$

will furnish a near approximation to the true value of y.

In fact, we have, from (1), $\quad y=\dfrac{.36}{y+6.6}$;

in which the first significant figure will be the same, whether we take $y=0$, or .09, as will be seen by dividing .36, successively, by 6.6, and by 6.69.

The same reasoning will apply, with still greater force, to the first figure of the root of $Z=0$.

Hence, we may find, in each instance, approximately, the next figure of the root by dividing the coefficient of y^0 and z^0 by the coefficient of y^1 and z^1.

The operation, then, will stand thus;

$$\begin{array}{lrrl}
1 & +5.8 & -5 & (.8541 \\
 & .8 & +4.64 & \\ \cline{2-3}
 & 6.65 & -.36 & \\
 & 5 & +.3325 & \\ \cline{2-3}
 & 6.704 & -.0275 & \\
 & 4 & +.026816 & \\ \cline{2-3}
 & 6.7081 & -.000684 & \\
 & 1 & +.00067081 & \\ \cline{2-3}
 & 6.7082 & -.00001319 &
\end{array}$$

§ 392. To explain this method of solution in a more general form, let a root of the equation,

$X = x^n + A_1x^{n-1} + A_2x^{n-2} \ldots + A_{n-1}x + A_n = 0,$
be $x = x' + y$; x' being the part of the root already found, and y representing the remaining figures, and y being, of course, very small compared with x' (§ 174. N. 1).

Then diminishing the roots of $X = 0$ by x', we have

$$Y = y^n + B_1y^{n-1} \ldots + B_{n-2}y^2 + B_{n-1}y + B_n = 0.$$

But, y being very small, its powers above the first may, for the moment, be neglected; and we shall have, nearly,

$$B_{n-1}y + B_n = 0;$$

or, also approximately, $y = -\dfrac{B_n}{B_{n-1}}$.

The correctness of the result will be verified by introducing into the transformed equation the figure so found.

Representing the figure so found by y', we shall have $y = y' + z$; and finding an equation, whose roots are less than those of $Y = 0$ by y', we shall, in like manner, find another figure of the root; and so on.

Hence, for finding a root of an equation of any degree whatever, we have the following

RULE.

§ 393. 1. *Find by Sturm's theorem, or by trial, the first figure, or the integral part, of the root.*

2. *Transform the equation into another, whose roots shall be less than those of the given equation by the part of the root already found.*

3. *With the last coefficient of the transformed equation for a dividend, and the last but one for a trial divisor, find the next figure of the root; and verify it by substitution in the transformed equation* (§ 350).

4. *Diminish the roots of the transformed equation by the figure just found, divide as before for the next figure; and so on, as far as is necessary.*

a.) The method is applicable to both positive and negative roots; each figure of a negative root being treated, in multiplying, as a negative quantity.

b.) A negative root is, however, more conveniently found by changing the signs of the alternate terms, and finding the corresponding positive root (§ 359).

§ 394. 1. Find the roots of the equation,

$$X = x^3 + 10x^2 + 5x - 260 = 0.$$

Here, $X' = 3x^2 + 20x + 5;$

$$X_2 = 17x + 239;$$

$$X_3 = -.$$

$x = -\infty$ gives *two* variations; $x = +\infty$, *one*. Hence, there is but one real root; positive, of course (§ 357. *b*).

We find, moreover, that the first figure of the positive root is 4.

1	+ 10	+ 5	− 260	(4.1179
	4	56	+ 244	
	14	61	,− 16	
	4	72	+ 13.521	
	18	,133	,− 2.479	
	4	2.21	1.376531	
	,22.1	135.21	,− 1.102469	
	1	2.22	.966221613	
	22.2	,137.43	,− .136247387	
	1	.2231	.124396356339	
	,22.31	137.6531	,− .011851030661	
	1	.2232		
	22.32	,137.8763		
	1	.155359		
	,22.337	138.031659		
	7	.156408		
	22.344	,138.198067		
	7	.2011671		
	,22.3519	138.21817371		

The coefficients of the successive transformed equations are marked with commas, the first coefficient in each being the same as in the primitive equation. Thus, we shall have

$$Y = y^3 + 22y^2 + 133y - 16 = 0;$$

$$Z = z^3 + 22.3z^2 + 137.43z - 2.479 = 0; \text{ and so on.}$$

NOTE. It will be observed, that the .1 added to 22, does not form a part of the coefficient of y^2, but was *added* to that coefficient in forming the next. A similar remark applies, of course, to the subsequent coefficients; and to the example of § 391, where .8 is most conveniently added to 5, by being written after it.

a.) The coefficients of the two last terms (B_{n-1}, and B_n, [§ 392]) in each of the transformed equations have *unlike signs*. This is as it should be, in finding a *positive* root.

For, suppose that the least real root of $X = 0$ is positive; and represent the part already found by x'.

Then B_n and B_{n-1} are what X and X' become, when x' is substituted for x. Therefore, x' being less than the least real root, B_n and B_{n-1} (i. e. $f(x')$ and $f'(x')$) must have *unlike* signs (§§ 373. *b*; 385).

b.) Similar reasoning will apply to any other positive root, provided x' differs from that root less than the next inferior root of $X' = 0$ does (§ 385. *a*). See *g*, *h*, below.

c.) In approximating to a *negative* root (§ 394. *a*), x' is greater than the root; and, of course, if it is less than the next greater root of $X' = 0$, B_n and B_{n-1} (i. e. $f(x')$ and $f'(x')$), must have *like* signs.

d.) If, having found the root, 4.1179, we divide X by $x - 4.1179$, we shall have an equation of the second degree, from which we may find the remaining roots (§ 353. *c*).

e.) Otherwise; we know that the coefficients of x^2 and x^0 in the given equation are respectively the sum, and product of the three roots with their signs changed. Also, the coefficients of x^1 and x^0 in the depressed equation will be the sum and product of the two remaining roots with their signs changed (§ 355. 1, 4).

Hence, if we *diminish* the coefficient of x^2, and *divide* the coefficient of x^0, in the given equation, by the root found taken with a contrary sign, we shall have the coefficients of x^1 and x^0 in the depressed equation. Thus,

$$x^2 + [10 - (-4.1179)]\,x - \frac{260}{-4.11791} = 0,$$

or

$$x^2 + 14.1179\,x + 63.1365 = 0,$$

will give the remaining roots of the equation, which, are, evidently, imaginary (§ 388. 6).

f.) When the roots are all real, it is frequently quite as convenient to find a second root from the given equation, in the same manner as the first; and then find the third by adding the two roots found to the coefficient of x^2, and changing the sign of the result (§§ 355. 1; 388. 3).

3. Find the roots of the equation,

$$X = x^3 - 7x + 7 = 0.$$

Here

$$X' = 3x^2 - 7;$$
$$X_2 = 2x - 3;$$
$$X_3 = +.$$

Hence, there are *three* real roots; one between -3 and -4, and two between 1 and 2. Also, the first two figures of the roots are -3.0, 1.3, and 1.6.

To find the greatest root, proceed thus.

1	0	-7	$+7$	$(1.69202147 = x.$
	1	1	-6	
	1	-6	$,+1$	
	1	2	-1.104	
	2	$,-4$	$,-.104$	
	1	2.16	.100809	
	,3.6	-1.84	$,-.003191$	
	.6	2.52		
	4.2	$,+.68$		

Find the other figures of the root. Also find the other root.

g.) There are here two roots of $X = 0$, and only one of $X' = 0$ (viz. 1.528), greater than 1.

Consequently, the substitution of 1 for x renders X positive and X' negative (§ 372); giving

$$B_n = f(x') = 1, \text{ and } B_{n-1} = f'(x') = -4.$$

Again, 1.6 being greater than the greatest root of $X' = 0$, and less than that of $X = 0$, renders X negative and X' positive (§ 372); giving

$$B_n = f(x') = -.104, \text{ and } B_{n-1} = f'(x') = +.68.$$

If we had substituted 1.5, B_{n-1} would have remained negative; because 1.5 is less than the greatest root of $X' = 0$.

Hence, if the sign of B_n changes, that of B_{n-1} should change also. See *a*, above.

h.) It may, however, not change at the same figure of the root, for that figure may be common to the next greater root of $X = 0$ and of $X' = 0$. This occurs in the greatest root of the following equation. See 4, below.

i.) To find the negative root, we change the signs of the alternate terms (§ 393. *b*).

1	− 0	− 7	− 7	(3.048 917
	3	9	6	
	3	2	,− 1	
	3	18	.814464	
	6	,20	,− .185536	
	3	.3616	.166382 592	
	,9.04	20.3616	,− .019153 408	
	4	.3632	.018791 228169	
	9.08	,20.7248	,− .000362 179831	
	4	.0730 24	208 873763	
	,9. 128	20.7978 24	,− .000153 306068	
	8	.0730 88	146 211615	
	9. 136	,20.8709 12	,− .000007 094453	
	8	82 3041		
	,9. 1449	20.879 14241		
	9	8 23122		
	9. 1458	20 8 8 737363	$x = -3.048\,917$.	

k.) We should evidently have obtained the same result, as far as we have carried the approximation, and with mnch less labor, if we had neglected all the figures on the right of the vertical lines in the several columns.

4. Find the roots of the equation,

$$X = x^3 + 11x^2 - 102x + 180 = 0 \text{ (§ 388. 3).}$$

The roots are 3.229 52, 3.213 127 7, and — 17.442 648 96.

The greatest root of $X' = 0$ is 3.2213. Consequently, 3.22 substituted for x, will render both X and X' negative. But 3.229 will render X negative, and X' positive. See *h* above.

5. Find the roots of the equation,

$$8x^3 - 6x - 1 = 0.$$

It is not necessary for the application of Sturm's theorem, or of this method of approximation, to reduce the equation as in § 389. *c*.

We shall find, that there are *three* real roots; one positive, and two negative; and that their initial figures are .9, —.1, and —.7.

The equation may be put under this form,

$$x^3 - \tfrac{3}{4}x - \tfrac{1}{8} = x^3 - .75x - .125 = 0.$$

To find the negative root, proceed as follows.

1	0.7	—.75	+.125	(.76 &c.
	.7	.49	—.182	
	1.4	—.26	,—.057	
	.7	.98	.050976	
	,2.16	,+.72	,—.006024	
		.1296		
		.8496		

The roots are —.766 04, —.1737, and .9397.

6. Find the real root of the equation $x^3 - 2 = 0$; i. e. find the cube root of 2. *Ans.* 1.259 921.

7. Find the roots of the equation $x^2 - 2 = 0$; i. e. find the square root of 2. *Ans.* ±1.414 213 6.

Note. It will be observed, that the solution of the third and fourth examples is equivalent to the processes of §§ 174, 179. The

method is, obviously, equally applicable to the extraction of roots of large numbers. The trial divisor, however, approximates, of course, most closely to the complete divisor, when the part of the root not yet found is very small.

8. What is the cube root of 3 442 951?

Ans. 151.

9. Find the roots of the equation,

$$x^4 - 12x^2 + 12x - 3 = 0.$$

1	0	-12	$+12$	-3	(2.8 &c.
	2	4	-16	-8	
	2	-8	-4	,-11	
	2	8	0	8.9856	
	4	0	,-4	,-2.0144	
	2	12	15.232		
	6	,12	11.232		
	2	7.04			
	,8.8	19.04			

Ans. 2.858 083, .606 018, .443 276 9, and —3.907 378.

Continue the operation, and find the other roots. The work may be greatly abridged by rejecting all but one decimal figure in the column of x^3, two or three in the column of x^2, three, four or five in that of x^1, and four, five, six or seven in that of x^0.

10. Find the roots of the equation,

$$x^3 - 2x - 5 = 0.$$

Ans. 2.094 55; the other roots are imaginary.

11. Find the roots of the equation,

$$x^4 - x^2 + 2x - 1 = 0.$$

Ans. 0.618, and —1.618; the others, imaginary.

12. Find a root of the equation,

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0.$$

Ans. $x = 8.414\,454\,7$.

13. What is the *fifth* root of 2? *Ans.* 1.148 699.

NOTE. This method of finding the real roots of any equation, if incommensurable, approximately, if commensurable, exactly, is sometimes called Horner's method.

RECURRING, OR RECIPROCAL EQUATIONS.

§ 395. The general form of a *recurring*, or *reciprocal* (§ 371) equation of an *odd* degree, is, obviously,

$$x^{2n+1}+A_1x^{2n}+A_2x^{2n-1}\ldots\pm A_2x^2\pm A_1x\pm 1=0;\quad (1)$$

in which the like coefficients belong, one to an even, and the other to an odd power of x throughout.

Now, if the corresponding coefficients have like signs, the substitution of -1, and, if they have unlike signs, the substitution of $+1$, for x, will render the corresponding terms numerically equal with contrary signs; and will, therefore, reduce the first member to 0. Hence,

One of the roots of a recurring equation of an *odd* degree is -1, or $+1$, according as the corresponding coefficients have *like* or *unlike* signs.

a.) Again, the equation may be written thus,

$$(x^{2n+1}\pm 1)+A_1x(x^{2n-1}\pm 1)+A_2x^2(x^{2n-3}\pm 1)\ldots=0;\quad (2)$$

in which $x=-1$, if we take the upper signs, and $x=+1$, if we take the lower signs, will render each of the quantities enclosed in parenthesis equal to zero.

b.) Let $2n+1=5$. Then the equation becomes

$$x^5+A_1x^4+A_2x^3\pm A_2x^2\pm A_1x\pm 1=0;\quad (3)$$

or $$(x^5\pm 1)+A_1x(x^3\pm 1)+A_2x^2(x\pm 1)=0.\quad (4)$$

Now, if we divide either the first member of (3) or each term of (4) by $x\pm 1$ (§§ 98, 96), taking always the upper signs together, and the lower signs together, we shall have

$$\left.\begin{array}{r} x^4\mp 1\\ +A_1\end{array}\right|\left.\begin{array}{r} x^3+1\\ \mp A_1\\ +A_2\end{array}\right|\left.\begin{array}{r} x^2\mp 1\\ +A_1\end{array}\right| x+1=0;\quad (5)$$

evidently an equation of an *even* degree (the $2n$th), whose coefficients at equal distances from the extremes are *equal* (i. e. are numerically equal and have like signs). It is, therefore, a *recurring* equation (§ 370. *b*).

The same reasoning will, obviously, apply to any similar equation as well as to that of the fifth degree.

§ 396. The general form of a recurring or reciprocal equation of an *even* degree, in which the like *coefficients* have *unlike signs* and the *middle* term is *wanting* (§ 370. *c*), is, obviously,

$$x^{2n+2} + A_1 x^{2n+1} \, . \, . + 0x^n - \, . \, . - A_1 x - 1 = 0. \quad (6)$$

Arranging (§ 34. *c*) according to the coefficients, we have

$$(x^{2n+2}-1) + A_1 x(x^{2n}-1) + A_2 x^2(x^{2n-2}-1) \, . \, . = 0\,; \quad (7)$$

each term of which is, evidently, divisible by x^2-1 (§ 96), i. e. by $(x+1)(x-1)$ [§ 93]. Hence,

A *recurring* equation of an *even* degree, in which the *middle term is wanting* and the corresponding coefficients have *unlike* signs, has its first member divisible by x^2-1; and, of course, has the two roots, -1 and $+1$.

a.) Let $2n+2=6$. Then the equation becomes

$$x^6 + A_1 x^5 + A_2 x^4 - A_2 x^2 - A_1 x - 1 = 0\,; \quad (8)$$

or

$$(x^6 - 1) + A_1 x(x^4 - 1) + A_2 x^2(x^2 - 1) = 0. \quad (9)$$

Now if we divide either the first member of (8) or each term of (9) by x^2-1, we shall have

$$x^4 + A_1 x^3 + \left.\begin{array}{r} A_2 \\ +1 \end{array}\right| x^2 + A_1 x + 1 = 0\,; \quad (10)$$

a *recurring* equation of an *even* degree, whose *like coefficients* have *like signs*, as in § 395. *b.*

b.) Otherwise; the roots of the depressed equations, (5) and (10), are the remaining roots of the primitive equations, (3) and (8); and one half of them are, therefore, the reciprocals of the other half.

§ 397. The general form of a recurring equation of an *even* degree, in which the *like coefficients* have *like signs*, is

$$x^{2n} + A_1 x^{2n-1} \, . \, . + A_n x^n \, . \, . + A_1 x + 1 = 0. \quad (11)$$

Dividing by x^n, we have

$$x^n + A_1 x^{n-1} \, . \, . \, . + A_{n-1} x + A_n + A_{n-1}\frac{1}{x} \, . \, . \, .$$

$$. \, . \, . \, . \, . + A_1 \frac{1}{x^{n-1}} + \frac{1}{x^n} = 0\,; \quad (12)$$

or

$$x^n+\frac{1}{x^n}+A_1\left(x^{n-1}+\frac{1}{x^{n-1}}\right)\,..\,+A_{n-1}\left(x+\frac{1}{x}\right)+A_n=0. \quad (13)$$

Now put $$x+\frac{1}{x}=z. \quad (a)$$

Then, squaring and transposing,

$$x^2+\frac{1}{x^2}=z^2-2. \quad (b)$$

Multiplying (b) by $$x+\frac{1}{x}=z,$$

$$x^3+\frac{1}{x^3}+x+\frac{1}{x}=z^3-2z\,; \text{ or } x^3+\frac{1}{x^3}=z^3-3z. \quad (c)$$

Or, in general, since

$$\left(x^m+\frac{1}{x^m}\right)\left(x+\frac{1}{x}\right)=x^{m+1}+\frac{1}{x^{m+1}}+x^{m-1}+\frac{1}{x^{m-1}}, \quad (d)$$

we have, by transposing,

$$x^{m+1}+\frac{1}{x^{m+1}}=\left(x^m+\frac{1}{x^m}\right)\left(x+\frac{1}{x}\right)-\left(x^{m-1}+\frac{1}{x^{m-1}}\right). \quad (e)$$

Thus, making $m=3$,

$$x^4+\frac{1}{x^4}=\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)-\left(x^2+\frac{1}{x^2}\right).$$

$\therefore$ from (a), (b), and (c),

$$x^4+\frac{1}{x^4}=(z^3-3z)z-(z^2-2)=z^4-4z^2+2. \quad (f)$$

Substituting these values of $x+x^{-1}$, x^2+x^{-2}, &c. in (13), we shall have an equation of the nth degree in z; i. e. of *half the degree* of the primitive equation, (11). Hence,

§ 398. A *recurring* equation of an *even* degree, in which the *like coefficients* have *like signs, can always be reduced to an equation of half that degree.*

a.) Hence (§§ 395, 396),

Cor. A *recurring* equation of *an odd* degree $(2n+1)$, or one of an *even* degree $(2n+2)$ whose *middle term is wanting* and whose *like coefficients* have *unlike signs, can always be reduced to an equation* of the nth degree.

b.) The solution of the equation of the nth degree gives

the values of z; and the values of x may be found from the equation,

$$x+\frac{1}{x}=z; \quad \text{i. e. } x^2-zx=-1.$$

1. Find the roots of the equation.

$$x^5-11x^4+17x^3+17x^2-11x+1=0.$$

One root is -1 (§ 395). Therefore, dividing by $x+1$ (§ 348, 350), we have

$$x^4-12x^3+29x^2-12x+1=0.$$

Dividing by x^2, $\left(x^2+\frac{1}{x^2}\right)-12\left(x+\frac{1}{x}\right)+29=0.$

Substituting,

$$z^2-2-12z+29=0; \text{ or } z^2-12z+27=0.$$

$$\therefore \qquad z=x+\frac{1}{x}=9, \text{ or } 3.$$

If $z=9$, $x^2-9x=-1$, and $x=\frac{1}{2}(9\pm\sqrt{77})$;
if $z=3$, $x^3-3x=-1$, and $x=\frac{1}{2}(3\pm\sqrt{5})$.

Therefore, the five roots are

$$-1, \frac{9+\sqrt{77}}{2}, \quad \frac{9-\sqrt{77}}{2}, \quad \frac{3+\sqrt{5}}{2}, \text{ and } \frac{3-\sqrt{5}}{2};$$

or, rendering the numerators of the third and fifth roots rational (§ 187),

$$-1, \frac{9+\sqrt{77}}{2}, \quad \frac{2}{9+\sqrt{77}}, \quad \frac{3+\sqrt{5}}{2}, \text{ and } \frac{2}{3+\sqrt{5}};$$

the third root being the reciprocal of the second, and the fifth, of the fourth (§ 120. *d*).

2. Find the roots of the equation,

$$4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0.$$

The reduced equation is

$$4z^3-24z^2+45z-25=0,$$

whose roots are 1, $\frac{5}{2}$ and $\frac{5}{2}$.

Hence, the roots of the given equation are

$$2, \tfrac{1}{2}, 2, \tfrac{1}{2}, \frac{1+\sqrt{-3}}{2}, \text{ and } \frac{1-\sqrt{-3}}{2}.$$

3. Solve the equation,

$$5x^4 + 8x^3 + 9x^2 + 8x + 5 = 0 \text{ (§ 388. N.)}.$$

Ans. $5z^2 + 8z - 1 = 0$.

$$x = .05825 \pm \sqrt{(-.9966)},$$
$$x = -.85825 \pm \sqrt{(-2634)}.$$

4. Solve the equation,

$$x^6 - 6\tfrac{3}{4}x^5 + 11\tfrac{5}{8}x^4 - 11\tfrac{5}{8}x^2 + 6\tfrac{3}{4}x^5 - 1 \text{ (§ 398. } a\text{)}.$$

The roots are 1, -1, 2, $\frac{1}{2}$, 4, and $\frac{1}{4}$.

BINOMIAL EQUATIONS.

§ 399. Equations of the form,

$$y^n \pm A = 0, \tag{1}$$

containing but *two terms*, are called BINOMIAL equations.

Suppose $A^{\frac{1}{n}} = a$, i. e. $A = a^n$.

Then we have $y^n \pm a^n = 0$.

Putting $y = ax$, $a^n x^n \pm a^n = 0$;

or
$$x^n \pm 1 = 0. \tag{2}$$

§ 400. I. Let n be an *odd* number, $2m + 1$.

Then
$$x^{2m+1} \pm 1 = 0, \tag{3}$$

being a recurring equation of an odd degree (§ 395), has *one real* root equal to -1, or $+1$, according as the last term is positive or negative.

1. Let the equation be $x^{2m+1} - 1 = 0. \quad (4)$

Then $+1$ is a root; and dividing by $x-1$, we have (§ 96)

$$x^{2m} + x^{2m-1} + x^{2m-2} \dots + x^2 + x + 1 = 0, \tag{5}$$

which can be reduced to an equation of the mth degree (§ 398).

Moreover, (4) has *no other real* root. For, if x be negative, x^{2m+1} will be negative (§ 151. c); and, if x be positive and different from 1, x^{2m+1} evidently cannot be equal to 1.

Consequently, all the roots of (5) are *imaginary*.

a.) This is evident, also, from the number ($2m$) of consecutive terms wanting in (4). See § 364. 2.

2. The equation, $$x^{2m+1}+1=0, \qquad (6)$$
has (§ 395) one real root equal to -1; and, reasoning as above, it is evident, that it can have no other real root.

If we divide by $x+1$ (§ 98), we shall have the equation
$$x^{2m}-x^{2m-1}+\ldots\ldots+x^2-x+1=0, \qquad (7)$$
containing the remaining roots, which can be reduced by § 397.

b.) Also, the roots of $x^{2m+1}+1=0$ are the same as those of $x^{2m+1}-1=0$, taken with contrary signs (§ 359).

§ 401. II. Again, let n be an *even* number, $2m$.

1. Then $$x^{2m}-1=0 \qquad (8)$$
has *two real* roots, $+1$ and -1 (§ 396).

It has also no other real roots. For, if we divide by x^2-1, we have
$$x^{2m-2}+x^{2m-4}\ldots\ldots+x^2+1=0; \qquad (9)$$
in which the powers of x being all *even* (§ 151. *c*), any real value of x, whether positive or negative, will render the first member positive (§ 358. 3), i. e. >0.

This equation can be reduced also to one of the $(m-1)$th degree (§ 398).

a.) Moreover, we have
$$x^{2m}-1=(x^m-1)(x^m+1=0.$$
$$\therefore \qquad x^m-1=0, \text{ and } x^m+1=0.$$

2. *All* the roots of the equation,
$$x^{2m}+1=0, \qquad (10)$$
i. e. $x^{2m}=-1$, are imaginary (§ 22. 2).

This equation can be reduced to one of the mth degree (§ 398).

b.) In each of the equations, (8) and (10), there is a deficiency of an odd number ($2m-1$) of consecutive terms.

Consequently (10) must contain at least $2m$ imaginary roots; and (8), at least $2m-2$ (§ 364. 1).

§ 402. Let the real roots be suppressed from the equation, $x^n \mp 1 = 0$; and let the equation in z, $Z=0$, be found (§ 397).

Let also one of the imaginary values of x be $a+b\sqrt{-1}$. Then we shall have

$$z = a + b\sqrt{-1} + \frac{1}{a+b\sqrt{-1}}.$$

But (§§ 187, 162)

$$\frac{1}{a+b\sqrt{-1}} = \frac{a-b\sqrt{-1}}{(a+b\sqrt{-1})(a-b\sqrt{-1})} = \frac{a-b\sqrt{-1}}{a^2+b^2}.$$

Moreover, if $a+b\sqrt{-1}$ be a root of the equation, $x^n \mp 1 = 0$, $a-b\sqrt{-1}$ must be a root also (§ 357).

Hence we shall have

$$(a+b\sqrt{-1})^n = \pm 1, \text{ and } (a-b\sqrt{-1})^n = \pm 1.$$

$$\therefore \quad [(a+b\sqrt{-1})(a-b\sqrt{-1})]^n = (a^2+b^2)^n = 1.$$

And since a^2+b^2 is a positive quantity, we have

$$a^2+b^2 = 1.$$

$$\therefore \qquad \frac{1}{a+b\sqrt{-1}} = a - b\sqrt{-1};$$

and $\qquad z = a+b\sqrt{-1} + a - b\sqrt{-1} = 2a.$

Hence, *all* the roots of the equation, $Z=0$, are *real*.

§ 403. Let a be one of the imaginary roots of the equation,

$$x^n - 1 = 0.$$

Then we have $\quad a^n = 1;\ a^{2n} = 1;\ a^{3n} = 1;$ &c.

also $\qquad a^{-n} = 1;\ a^{-2n} = 1;\ a^{-3n} = 1;$ &c. Hence,

If a be an *imaginary root* of the equation $x^n - 1 = 0$, then will *any integral power* of a be a root also.

a.) As the equation can have but n roots, many of these powers of a must be equal to one another.

Thus, the imaginary roots of $x^4 - 1 = 0$ are $+\sqrt{-1}$ and $-\sqrt{-1}$. Now we have (§ 162)

$(\sqrt{-1})^2 = -1$; $(\sqrt{-1})^3 = -\sqrt{-1}$; $(\sqrt{-1})^4 = 1$;
$(\sqrt{-1})^5 = \sqrt{-1}$; $(\sqrt{-1})^6 = -1$; $(\sqrt{-1})^7 = -\sqrt{-1}$;

b.) It must be understood, however, that these are only different ways of expressing the same roots. The equation, $x^n \mp 1 = 0$, has no equal roots; since its derived equation $nx^{n-1} = 0$ has no common measure with it (§ 378. *f*).

§ 404. Let a be an imaginary root of the equation,

$$x^n + 1 = 0.$$

Then we have

$a^n = -1$; $(a^n)^3 = (a^3)^n = -1$; $(a^n)^5 = (a^5)^n = -1$; also $(a^n)^{-3} = (a^{-3})^n = -1$; $(a^{2m+1})^n = -1$. Hence,

If a be an *imaginary root* of the equation, $x^n + 1 = 0$, then will *any odd integral power* of a be a root also.

Thus, the imaginary roots of $x^2 + 1 = 0$ are $+\sqrt{-1}$ and $-\sqrt{-1}$; and all the odd integral powers of either of these roots are also roots (§ 403. *a*).

§ 405. Find the roots of the following equations;

1. $x^3 - 1 = 0$.

Ans. 1, $\dfrac{-1+\sqrt{-3}}{2}$, and $\dfrac{-1-\sqrt{-3}}{2}$.

2. $x^4 - 1 = 0$. *Ans.* 1, -1, $\sqrt{-1}$, and $-\sqrt{-1}$.

3. $x^6 - 1 = 0$; i. e. $(x^3 - 1)(x^3 + 1) = 0$.

Ans. 1, -1, $\dfrac{1 \pm \sqrt{-3}}{2}$, and $\dfrac{-1 \pm \sqrt{-3}}{2}$.

NOTE. The roots of the equations, $x^2 - 1 = 0$, $x^3 - 1 = 0$, &c., are sometimes called the *roots of unity*. It is evident (§§ 151. *a*; 152. *a*), that the roots of any other number, of any degree, may be found by multiplying one of them, most conveniently, the arithmetical root, by the several roots of unity of the same degree.

CHAPTER XVII.

CONTINUED FRACTIONS.

§ 406. A CONTINUED FRACTION *is one whose numerator is a whole number, and whose denominator is a whole number plus a fraction, which also has a whole number for its numerator, and for its denominator a whole number plus a fraction; and so on.*

We shall consider only those, in which each of the numerators is unity, and the partial denominators (a, below) are all positive. Thus,

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\&c.}}}, \quad (1) \qquad \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\&c.}}} \quad (2)$$

are continued fractions.

a.) The integral parts of the denominators are sometimes called *partial* denominators, or *partial quotients*; and the fractions, $\frac{1}{2}$, $\frac{1}{3}$, &c., $\frac{1}{a_1}$, $\frac{1}{a_2}$, &c., are called *partial*, or *integral* fractions.

§ 407. If, in (1) above, we neglect all but the *first* partial fraction, the denominator 2 will be less than the true denominator; and, of course, $\frac{1}{2}$ is *greater* than the true value of the continued fraction.

Again, suppose we neglect all but *two* partial fractions. Then, the partial denominator, 3, being too small, the par-

tial fraction, $\frac{1}{3}$, is too great; and, consequently, $2\frac{1}{3}$ being greater than the true denominator, the fraction,

$$\frac{1}{2+\frac{1}{3}}=\frac{1}{\frac{7}{3}}=\frac{3}{7},$$

will be *less* than the true value of the continued fraction.

Similar reasoning will, evidently, hold in respect to any number of terms; and will apply equally to the general form (2), as to the particular example we have considered.

Hence,

If we include in the reduction an ODD *number of partial fractions, the result will be too* GREAT; *if an* EVEN *number, the result will be too* SMALL.

a.) The fractions, $\frac{1}{a_1}$, $\quad \frac{1}{a_1+\frac{1}{a_2}}$, $\quad \frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3}}}$, &c.,

are *approximate values* of the given fraction; and are sometimes called *approximating* or *converging* fractions, or simply, *convergents.*

b.) It is evident, that the *true* value of the continued fraction, lying between two successive *approximate* values, *differs from either of them less than they differ from each other.*

§ 408. We have $\frac{1}{a_1}=\frac{1}{a_1}$, 1st approx. value.

$$\frac{1}{a_1+\frac{1}{a_2}}=\frac{a_2}{a_1a_2+1}, \quad \text{2d} \quad \text{“} \quad \text{“}$$

$$\frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3}}}=\frac{a_2+\frac{1}{a_3}}{a_1\left(a_2+\frac{1}{a_3}\right)+1}$$

$$=\frac{a_2a_3+1}{(a_1a_2+1)a_3+a_1}, \quad \text{3d} \quad \text{“} \quad \text{“}$$

We shall, evidently, find the *fourth* approximate value, or convergent, by substituting, in the *third*, $a_3 + \frac{1}{a_4}$ for a_3.

Thus,

$\frac{(a_2 a_3 + 1)a_4 + a_2}{[(a_1 a_2 + 1)a_3 + a_1]a_4 + a_1 a_2 + 1}$ is the *fourth* convergent.

We find, obviously, the *numerator and denominator of the third convergent, by multiplying those of the second by the third partial denominator, and adding those of the first convergent.*

We find, in like manner, the fourth convergent from the terms of the second and third.

To show the generality of this law, let it be admitted to hold good as far as the nth convergent (i. e. the convergent corresponding to a_n).

Let also $\frac{L}{L'}$, $\frac{M}{M'}$, $\frac{N}{N'}$, and $\frac{P}{P'}$ be the convergents corresponding to a_{n-2}, a_{n-1}, a_n, and a_{n+1}.

Then, since the nth convergent is formed according to the above law, we shall have

$$\frac{N}{N'} = \frac{Ma_n + L}{M'a_n + L'}. \tag{3}$$

If now we substitute in $\frac{N}{N'}$, $a_n + \frac{1}{a_{n+1}}$ for a_n, we shall, obviously, find $\frac{P}{P'}$. Thus,

$$\frac{P}{P'} = \frac{M\left(a_n + \frac{1}{a_{n+1}}\right) + L}{M'\left(a_n + \frac{1}{a_{n+1}}\right) + L'} = \frac{(Ma_n + L)a_{n+1} + M}{(M'a_n + L')a_{n+1} + M'};$$

or, from (3),

$$\frac{P}{P'} = \frac{Na_{n+1} + M}{N'a_{n+1} + M'}. \tag{4}$$

Consequently, if the law holds good for n convergents, it will for $n + 1$.

Hence, to find the numerator and denominator of any

convergent after the second, as the $(n+1)$th, we have the following

RULE.

§ 409. *Multiply the numerator and denominator of the nth convergent by the $(n+1)$th partial denominator, and add to the products, respectively, the numerator and denominator of the $(n-1)$th convergent.*

a.) The numerator and denominator of any convergent must be respectively greater than those of the preceding; each numerator and each denominator being at least equal to the sum of the two next preceding.

b.) Moreover, each convergent is found by substituting, in the preceding, for the last *partial* denominator, an expression known to approach more nearly to the *true* denominator.

Hence, evidently, each convergent approximates more closely than the preceding to the true value of the continued fraction. See § 410. *a*, *b*.

1. Find the successive convergents of the continued fraction,

$$\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{87}}}}}.$$

Ans. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{4}{11}$, and $\frac{351}{965}$.

c.) The first four convergents are *approximate* values of the continued fraction; the last, $\frac{351}{965}$, is the *true* value.

d.) A continued fraction is sometimes *mixed* (§ 112), i. e. made up of a whole number and a fraction. Thus,

$$3+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\&c.}}} \qquad (5)$$

In such cases, the integral part may be reserved and added to the convergents; or it may be taken, with 1 as a denominator, for the first convergent.

Thus, in the above example, we shall have the convergents, $3\frac{1}{2}$, $3\frac{3}{7}$, $3\frac{16}{37}$; or $\frac{3}{1}$, $\frac{7}{2}$, $\frac{24}{7}$, $\frac{127}{37}$.

e.) This form,
$$a+\cfrac{1}{a_1+\cfrac{1}{a_2+\&c.,}} \tag{6}$$
is sometimes assumed as the general form of a continued fraction; the place of the integral part, when it is wanting, being filled with 0.

In that case, the *first* convergent is, evidently, too *small;* the *second*, too *great;* and so on, those of an *odd* order being too *small*, and those of an *even* order, too *great*. See § 407.

NOTE. If the integral part be *zero*, the first convergent will of course be $\frac{0}{1}$.

§ 410. If the second convergent of § 408 be subtracted from the first, the remainder is *unity* divided by the product of the denominators. If the third be subtracted from the second, the remainder is *minus unity* divided by the product of the denominators.

Suppose it has been proved, that this law extends to $n-1$ convergents; i. e. that
$$\frac{L}{L'}-\frac{M}{M'}=\frac{LM'-L'M}{L'M'}=\frac{\pm 1}{L'M'}. \tag{7}$$
Then $\frac{M}{M'}-\frac{N}{N'}=\frac{M}{M'}-\frac{Ma_n+L}{M'a_n+L'}$
$$=\frac{L'M-LM'}{M'N'}=-\frac{LM'-L'M}{M'N'}; \tag{8}$$
the numerator of which is the same as that of (7), with a contrary sign. Hence, the principle proved in regard to the first three convergents, applies equally to the whole series. That is,

If each convergent be subtracted from that which next

precedes, the numerator of the difference will be ± 1; *and the denominator will be the product of the denominators of the two convergents.*

a.) Again, the true value of the continued fraction lies between any two successive convergents, and differs from either of them less than they differ from each other (§ 407).

That is, the convergent $\frac{M}{M'}$, differs from the true value of the continued fraction by less than $\frac{1}{M'N'}$.

But (§ 409. *a*) $M' < N'$; and $\therefore$ $M'^2 < M'N'$.

$$\therefore \qquad \frac{1}{M'N'} < \frac{1}{M'^2}. \qquad \text{Hence,}$$

Cor. I. The *error*, in taking any convergent whatever for the true value of the continued fraction, is *numerically less than unity divided by the square of the denominator of that convergent.*

b.) The denominator of each convergent is greater than the next preceding by some whole number (§ 409. *a*).

Hence, if the fraction be infinite, we may find a convergent whose denominator shall be greater than any given quantity; and, consequently,

Cor. II. We may find a convergent, which shall differ from the true value of the continued fraction by *less than any given quantity.*

c.) Suppose that M and M' have a common divisor, D.

Then D will, of course, divide $L'M$ and LM', multiples of M and M'; and, consequently (§ 102. Note *c*), the difference of those multiples, $LM' - L'M = \pm 1$.

Therefore D must divide ± 1, which has no integral divisor but unity.

$$\therefore \qquad D = 1. \qquad \text{That is,}$$

Cor. III. *Every convergent is in its lowest terms.*

§ 411. One of the most obvious uses of continued fractions is, to express approximately, in small numbers, fractions whose terms are large. Thus,

1. $$\frac{17}{59}=\frac{1}{\left(\frac{59}{17}\right)}=\frac{1}{3+\frac{8}{17}}=\frac{1}{3+\frac{1}{\left(\frac{17}{8}\right)}}=\frac{1}{3+\frac{1}{2+\frac{1}{8}}}.$$

Here we first divide both numerator and denominator (§ 113. 3) of $\frac{17}{59}$ by 17. We then reduce $\frac{59}{17}$ to a mixed number $3\frac{8}{17}$; and, again, divide both terms of $\frac{8}{17}$ by 8, and reduce to a mixed number; and so on.

Evidently, these operations produce no change in the value of the given fraction.

a.) Now the several convergents of the continued frac- found, are $\frac{1}{3}$, $\frac{2}{7}$, and $\frac{17}{59}$.

We find $$\frac{1}{3}=\frac{19\frac{2}{3}}{59}, \text{ too } \textit{great;}$$

$$\frac{2}{7}=\frac{16\frac{6}{7}}{59}, \text{ too } \textit{small}, \text{ but differing from}$$

the true value by only $\frac{1}{413}$.

2. If the fraction proposed had been $\frac{59}{17}$, we should have found

$$\frac{59}{17}=3+\frac{8}{17}=3+\frac{1}{\frac{17}{8}}=3+\frac{1}{2+\frac{1}{8}};$$

and the convergents, 3, $\frac{7}{2}$, and $\frac{59}{17}$. § 409. *d.*

b.) This reduction of a common, to a continued fraction, is, evidently, effected by applying to the terms of the given fraction the *process of finding the greatest common divisor;* the several *quotients* forming the successive *partial denominators.*

§ 412. If it be required to transform any quantity whatever, x, into a continued fraction, the nature of continued fractions will sufficiently indicate the following

RULE.

1. *Find the greatest integer contained in x, and denote*

it by a; and denote the fractional excess of x above a by $\frac{1}{x_1}$. Then $x = a + \frac{1}{x_1}$. $\therefore x_1 = \frac{1}{x - a} > 1$.

2. *Find the greatest integer contained in x_1, and denote it by a_1; and denote the fractional excess of x_1 above a_1 by* $\frac{1}{x_2}$. Then $$x_1 = a_1 + \frac{1}{x_2}.$$

3. *Apply the same process to x_2, and so on.*

Thus,

$$x = a + \frac{1}{x_1} = a + \cfrac{1}{a_1 + \cfrac{1}{x_2}} = a + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}}, \text{ \&c.}$$

a.) If $x < 1$, we shall have $a = 0$.

b.) We shall *always* have x_1, x_2, &c. > 1.

For if $x_1 =$, or < 1, we have $\frac{1}{x_1} =$ or > 1; and a is not the greatest integer contained in x.

c.) Whenever we find a denominator, x_n, equal to a whole number, we shall have $x_n = a_n$; and the continued fraction will terminate.

This will happen, if the quantity, x, can be exactly expressed by a common fraction.

d.) If the quantity is not equal to a common fraction (i. e. if it is incommensurable), the continued fraction will extend to infinity.

§ 413. 1. Given $\pi = 3.14159$ (§ 247. N. q), employing only five decimal places. Reduce π to a continued fraction, and find approximate values.

$$\textit{Ans. } \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{25}}}} \text{ \&c.}$$

Convergents (§ 409. e), 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, &c.

NOTE. The second approximate value, $\frac{22}{7}$, was found by Archimedes; the fourth, $\frac{355}{113}$, by Adrian Metius.

2. The common, or tropical year consists of 365.242 241 mean solar days. Find approximate values for this time.

Ans. $365\frac{1}{4}$, $365\frac{7}{29}$, $365\frac{8}{33}$, $365\frac{39}{161}$, &c.

NOTE. The third approximation shows an excess of the solar year above 365 days, of $\frac{8}{33}$ of a day. To preserve the coincidence between the solar and civil year, therefore, eight years in thirty-three must contain 366 days each. That is, a day must be added to every fourth year seven times in succession, and, the eighth time, to the fifth year.

3. The sidereal month (i. e. the time of the moon's sidereal revolution) consists of 27.321 661 days; or, the moon revolves 1 000 000 times in 27 321 661 days. Find approximate values of this ratio. *Ans.* 27, $\frac{82}{3}$, $\frac{765}{28}$, $\frac{3907}{143}$, &c.

NOTE. These ratios show that the moon revolves about 3 times in 82 days; 28 times in 765 days; or, more exactly, 143 times in 3907 days.

§ 414. Continued fractions are also employed in finding the roots of equations, and in extracting the roots of numbers.

1. Extract the square root of 3; i. e. find a root of the equation,

$$x^2 - 3 = 0. \tag{1}$$

Here

$$x = 1 + \frac{1}{x_1}.$$

Diminishing the roots of (1) by 1 (§ 367), we have

$$y^2 + 2y - 2 = 0, \tag{2}$$

an equation, whose roots are equal to $\frac{1}{x_1}$.

Transforming (2) by § 370, we find

$$2x_1^2 - 2x_1 - 1 = 0. \tag{3}$$

This gives

$$x_1 = 1 + \frac{1}{x_2}.$$

Transforming (3) in the same manner as (1), we have

$$x_2^2 - 2x_2 - 2 = 0; \qquad (4) \qquad \text{and } x_2 = 2 + \frac{1}{x_3}.$$

We find, in like manner,

$$2x_3{}^2 - 2x_3 - 1 = 0, \qquad (5)$$

which being the same as (3), will have the same roots, and will give rise to transformed equations like (4) and (5).

Hence, we shall have a repetition of the equations (3) and (4), and of their roots of which 1 and 2 are the integral parts, in endless succession.

$$\therefore \qquad x = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{2} \text{ \&c.}}}} = 1.732 \text{ \&c.}$$

The convergents are 1, 2, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}$, $\frac{71}{41}$, $\frac{97}{56}$.

a.) A continued fraction of this kind, in which any number of the partial denominators are continually repeated in the same order, is called *periodic.*

b.) It will be found, that every incommensurable root of an equation of the second degree may be expressed by a periodic continued fraction.

Of course, when the first period is found, such a fraction may be developed to any extent, by simply repeating the period.

2. Extract the square root of 2.

Convergents, 1, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, $\frac{99}{70}$, &c.

ERRATA.

Page 14, first line, for "16" read "11."
" 60, line 27, for "$+b'$," read "$-b'$."
" 62, line 19, for "58," read "28."
" 69, last line, for "$+b$" read "$+ab$."
" 83, first line, for "$+b^3$" read "$-b^3$."
" 91, line 15, for "$a=b$" read "$b=a$;" and for "a^n-b^n" read "a^n-a^n."
" 92, " 20, for "$+ab$" read "$-ab$."
" 93, " 27, after "*them*" insert "*taken as a divisor.*"
" 96, " 26, for "$5ab^3$" read "$5a^3b$."
" 99, " 30, and page 100, line 26, for "12" read "11."
" 106, " 17, for "dividing" read "multiplying."
" 119, " 5, for "$13\frac{7}{31}$" read "$23\frac{7}{31}$."
" 128, " 25, for "(2.3^2)" read "$(2.3^2)^{\frac{1}{3}}$."
" 148, " 11, for "$a-b$" read "a^2-b."
" 163, " 4, for "$\sqrt{(p^2\mp q^2)}$" read "$\sqrt{(p^2-q^2)}$."
" 192, " 3, for "10" read "11."
" 198, " 3, for "(1)" read "(3)."
" 200, " 20 and 21, for "222 &c.," and "743 &c.," read ".222 &c.," and ".743 &c."
" 228, last line at the end, for "D_3" read "$\frac{(n-4)}{.4}D_4$."
" 266, line 15, for "$A_{n-1}a$" read "$A_{n-1}x$."
" 292, " 14, after "have," insert "(x being >1)."
" 310, " 11, for "+" read "—."